CHAPTER 9
Sequences, Series, and Probability

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Section 9.1 Sequences and Series

- Given the general nth term in a sequence, you should be able to find, or list, some of the terms.
- You should be able to find an expression for the apparent nth term of a sequence.
- You should be able to use and evaluate factorials.
- You should be able to use summation notation for a sum.
- You should know that the sum of the terms of a sequence is a series.

Vocabulary Check

1. infinite sequence
2. terms
3. finite
4. recursively
5. factorial
6. summation notation
7. index; upper; lower
8. series
9. nth partial sum

1. $a_n = 3n + 1$

$a_1 = 3(1) + 1 = 4$

$a_2 = 3(2) + 1 = 7$

$a_3 = 3(3) + 1 = 10$

$a_4 = 3(4) + 1 = 13$

$a_5 = 3(5) + 1 = 16$

2. $a_n = 5n - 3$

$a_1 = 5(1) - 3 = 2$

$a_2 = 5(2) - 3 = 7$

$a_3 = 5(3) - 3 = 12$

$a_4 = 5(4) - 3 = 17$

$a_5 = 5(5) - 3 = 22$

3. $a_n = 2^n$

$a_1 = 2^1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$a_4 = 2^4 = 16$

$a_5 = 2^5 = 32$

4. $a_n = \left(\frac{1}{2}\right)^n$

$a_1 = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$

$a_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$a_3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$a_4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

$a_5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

5. $a_n = (-2)^n$

$a_1 = (-2)^1 = -2$

$a_2 = (-2)^2 = 4$

$a_3 = (-2)^3 = -8$

$a_4 = (-2)^4 = 16$

$a_5 = (-2)^5 = -32$

6. $a_n = \left(-\frac{1}{2}\right)^n$

$a_1 = \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$

$a_2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

$a_3 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$

$a_4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16}$

$a_5 = \left(-\frac{1}{2}\right)^5 = -\frac{1}{32}$

7. $a_n = \dfrac{n+2}{n}$

$a_1 = \dfrac{1+2}{1} = 3$

$a_2 = \dfrac{4}{2} = 2$

$a_3 = \dfrac{5}{3}$

$a_4 = \dfrac{6}{4} = \dfrac{3}{2}$

$a_5 = \dfrac{7}{5}$

8. $a_n = \dfrac{n}{n+2}$

$a_1 = \dfrac{1}{1+2} = \dfrac{1}{3}$

$a_2 = \dfrac{2}{2+2} = \dfrac{1}{2}$

$a_3 = \dfrac{3}{3+2} = \dfrac{3}{5}$

$a_4 = \dfrac{4}{4+2} = \dfrac{2}{3}$

$a_5 = \dfrac{5}{5+2} = \dfrac{5}{7}$

9. $a_n = \dfrac{6n}{3n^2 - 1}$

$a_1 = \dfrac{6(1)}{3(1)^2 - 1} = 3$

$a_2 = \dfrac{6(2)}{3(2)^2 - 1} = \dfrac{12}{11}$

$a_3 = \dfrac{6(3)}{3(3)^2 - 1} = \dfrac{9}{13}$

$a_4 = \dfrac{6(4)}{3(4)^2 - 1} = \dfrac{24}{47}$

$a_5 = \dfrac{6(5)}{3(5)^2 - 1} = \dfrac{15}{37}$

10. $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$

$a_1 = \dfrac{3(1)^2 - 1 + 4}{2(1)^2 + 1} = 2$

$a_2 = \dfrac{3(2)^2 - 2 + 4}{2(2)^2 + 1} = \dfrac{14}{9}$

$a_3 = \dfrac{3(3)^2 - 3 + 4}{2(3)^2 + 1} = \dfrac{28}{19}$

$a_4 = \dfrac{3(4)^2 - 4 + 4}{2(4)^2 + 1} = \dfrac{16}{11}$

$a_5 = \dfrac{3(5)^2 - 5 + 4}{2(5)^2 + 1} = \dfrac{74}{51}$

11. $a_n = \dfrac{1 + (-1)^n}{n}$

$a_1 = 0$

$a_2 = \dfrac{2}{2} = 1$

$a_3 = 0$

$a_4 = \dfrac{2}{4} = \dfrac{1}{2}$

$a_5 = 0$

12. $a_n = 1 + (-1)^n$

$a_1 = 1 + (-1)^1 = 0$

$a_2 = 1 + (-1)^2 = 2$

$a_3 = 1 + (-1)^3 = 0$

$a_4 = 1 + (-1)^4 = 2$

$a_5 = 1 + (-1)^5 = 0$

13. $a_n = 2 - \dfrac{1}{3^n}$

$a_1 = 2 - \dfrac{1}{3} = \dfrac{5}{3}$

$a_2 = 2 - \dfrac{1}{9} = \dfrac{17}{9}$

$a_3 = 2 - \dfrac{1}{27} = \dfrac{53}{27}$

$a_4 = 2 - \dfrac{1}{81} = \dfrac{161}{81}$

$a_5 = 2 - \dfrac{1}{243} = \dfrac{485}{243}$

14. $a_n = \dfrac{2^n}{3^n}$

$a_1 = \dfrac{2^1}{3^1} = \dfrac{2}{3}$

$a_2 = \dfrac{2^2}{3^2} = \dfrac{4}{9}$

$a_3 = \dfrac{2^3}{3^3} = \dfrac{8}{27}$

$a_4 = \dfrac{2^4}{3^4} = \dfrac{16}{81}$

$a_5 = \dfrac{2^5}{3^5} = \dfrac{32}{243}$

15. $a_n = \dfrac{1}{n^{3/2}}$

$a_1 = \dfrac{1}{1} = 1$

$a_2 = \dfrac{1}{2^{3/2}}$

$a_3 = \dfrac{1}{3^{3/2}}$

$a_4 = \dfrac{1}{4^{3/2}} = \dfrac{1}{8}$

$a_5 = \dfrac{1}{5^{3/2}}$

16. $a_n = \dfrac{10}{n^{2/3}} = \dfrac{10}{\sqrt[3]{n^2}}$

$a_1 = \dfrac{10}{1} = 10$

$a_2 = \dfrac{10}{\sqrt[3]{2^2}} = \dfrac{10}{\sqrt[3]{4}}$

$a_3 = \dfrac{10}{\sqrt[3]{3^2}} = \dfrac{10}{\sqrt[3]{9}}$

$a_4 = \dfrac{10}{\sqrt[3]{4^2}} = \dfrac{10}{\sqrt[3]{16}}$

$a_5 = \dfrac{10}{\sqrt[3]{5^2}} = \dfrac{10}{\sqrt[3]{25}}$

17. $a_n = \dfrac{(-1)^n}{n^2}$

$a_1 = -\dfrac{1}{1} = -1$

$a_2 = \dfrac{1}{4}$

$a_3 = -\dfrac{1}{9}$

$a_4 = \dfrac{1}{16}$

$a_5 = -\dfrac{1}{25}$

18. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$

$a_1 = (-1)^1\dfrac{1}{1+1} = -\dfrac{1}{2}$

$a_2 = (-1)^2\dfrac{2}{2+1} = \dfrac{2}{3}$

$a_3 = (-1)^3\dfrac{3}{3+1} = -\dfrac{3}{4}$

$a_4 = (-1)^4\dfrac{4}{4+1} = \dfrac{4}{5}$

$a_5 = (-1)^5\dfrac{5}{5+1} = -\dfrac{5}{6}$

19. $a_n = \frac{2}{3}$

$a_1 = \frac{2}{3}$

$a_2 = \frac{2}{3}$

$a_3 = \frac{2}{3}$

$a_4 = \frac{2}{3}$

$a_5 = \frac{2}{3}$

20. $a_n = 0.3$

$a_1 = 0.3$

$a_2 = 0.3$

$a_3 = 0.3$

$a_4 = 0.3$

$a_5 = 0.3$

21. $a_n = n(n-1)(n-2)$

$a_1 = (1)(0)(-1) = 0$

$a_2 = (2)(1)(0) = 0$

$a_3 = (3)(2)(1) = 6$

$a_4 = (4)(3)(2) = 24$

$a_5 = (5)(4)(3) = 60$

22. $a_n = n(n^2 - 6)$

$a_1 = 1(1^2 - 6) = -5$

$a_2 = 2(2^2 - 6) = -4$

$a_3 = 3(3^2 - 6) = 9$

$a_4 = 4(4^2 - 6) = 40$

$a_5 = 5(5^2 - 6) = 95$

23. $a_{25} = (-1)^{25}(3(25) - 2) = -73$

24. $a_n = (-1)^{n-1}[n(n-1)]$

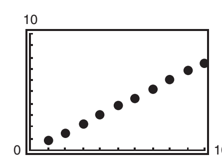
$a_{16} = (-1)^{16-1}[16(16-1)] = -240$

25. $a_{11} = \dfrac{4(11)}{2(11)^2 - 3} = \dfrac{44}{239}$

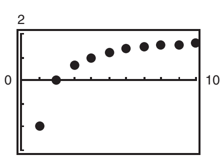
26. $a_n = \dfrac{4n^2 - n + 3}{n(n-1)(n+2)}$

$a_{13} = \dfrac{4(13)^2 - 13 + 3}{13(13-1)(13+2)} = \dfrac{37}{130}$

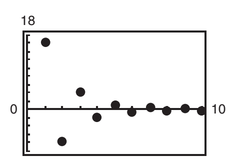
27. $a_n = \dfrac{3}{4}n$



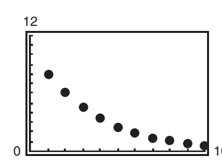
28. $a_n = 2 - \dfrac{4}{n}$



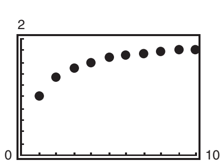
29. $a_n = 16(-0.5)^{n-1}$



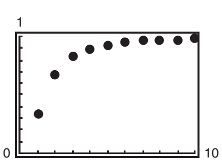
30. $a_n = 8(0.75)^{n-1}$



31. $a_n = \dfrac{2n}{n+1}$



32. $a_n = \dfrac{n^2}{n^2 + 2}$



33. $a_n = \dfrac{8}{n+1}$

$a_1 = 4,\ a_{10} = \dfrac{8}{11}$

The sequence decreases.

Matches graph (c).

34. $a_n = \dfrac{8n}{n+1}$

$a_n \to 8$ as $n \to \infty$

$a_1 = 4, \quad a_3 = \dfrac{24}{4} = 6$

Matches graph (b).

35. $a_n = 4(0.5)^{n-1}$

$a_1 = 4, a_{10} = \dfrac{1}{128}$

The sequence decreases.

Matches graph (d).

36. $a_n = \dfrac{4^n}{n!}$

$a_n \to 0$ as $n \to \infty$

$a_1 = 4, a_4 = \dfrac{4^4}{4!} = \dfrac{256}{24} = 10\dfrac{2}{3}$

Matches graph (a).

37. $1, 4, 7, 10, 13, \ldots$

$a_n = 1 + (n-1)3 = 3n - 2$

38. $3, 7, 11, 15, 19, \ldots$

n:	1	2	3	4	5	. . .	n
Terms:	3	7	11	15	19	. . .	a_n

Apparent pattern:

Each term is one less than four times n, which implies that $a_n = 4n - 1$.

39. $0, 3, 8, 15, 24, \ldots$

$a_n = n^2 - 1$

40. $2, -4, 6, -8, 10, \ldots$

n:	1	2	3	4	5	. . .	n
Terms:	2	-4	6	-8	10	. . .	a_n

Apparent pattern:

Each term is the product of $(-1)^{n+1}$ and twice n, which implies that $a_n = (-1)^{n+1}(2n)$.

41. $-\dfrac{2}{3}, \dfrac{3}{4}, -\dfrac{4}{5}, \dfrac{5}{6}, -\dfrac{6}{7}, \ldots$

$a_n = (-1)^n\left(\dfrac{n+1}{n+2}\right)$

42. $\dfrac{1}{2}, \dfrac{-1}{4}, \dfrac{1}{8}, \dfrac{-1}{16}, \ldots$

n:	1	2	3	4	. . .	n
Terms:	$\frac{1}{2}$	$\frac{-1}{4}$	$\frac{1}{8}$	$\frac{-1}{16}$	. . .	a_n

Apparent pattern:

Each term is $(-1)^{n+1}$ divided by 2 raised to the n, which implies that

$a_n = \dfrac{(-1)^{n+1}}{2^n}$.

43. $\dfrac{2}{1}, \dfrac{3}{3}, \dfrac{4}{5}, \dfrac{5}{7}, \dfrac{6}{9}, \ldots$

$a_n = \dfrac{n+1}{2n-1}$

44. $\dfrac{1}{3}, \dfrac{2}{9}, \dfrac{4}{27}, \dfrac{8}{81}, \ldots$

n:	1	2	3	4	. . .	n
Terms:	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{4}{27}$	$\frac{8}{81}$	. . .	a_n

Apparent pattern:

Each term is 2^{n-1} divided by 3 raised to the n, which implies that

$a_n = \dfrac{2^{n-1}}{3^n}$.

45. $1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \dfrac{1}{25}, \ldots$

$a_n = \dfrac{1}{n^2}$

46. $1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \ldots$

n:	1	2	3	4	5	. . .	n
Terms:	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	. . .	a_n

Apparent pattern:

Each term is the reciprocal of $n!$, which implies that

$a_n = \dfrac{1}{n!}$.

47. $1, -1, 1, -1, 1, \ldots$

$a_n = (-1)^{n+1}$

48. $1, 2, \frac{2^2}{2}, \frac{2^3}{6}, \frac{2^4}{24}, \frac{2^5}{120}, \ldots$

n:	1	2	3	4	5	6	. . .	n
Terms:	1	2	$\frac{2^2}{2}$	$\frac{2^3}{6}$	$\frac{2^4}{24}$	$\frac{2^5}{120}$	. . .	a_n

Apparent pattern:
Each term is 2^{n-1} divided by $(n-1)!$, which implies that

$$a_n = \frac{2^{n-1}}{(n-1)!}.$$

49. $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

$$a_n = 1 + \frac{1}{n}$$

50. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

n:	1	2	3	4	5	. . .	n
Terms:	$1 + \frac{1}{2}$	$1 + \frac{3}{4}$	$1 + \frac{7}{8}$	$1 + \frac{15}{16}$	$1 + \frac{31}{32}$	. . .	a_n

Apparent pattern: Each term is the sum of 1 and the quantity 1 less than 2^n divided by 2^n, which implies that

$$a_n = 1 + \frac{2^n - 1}{2^n}.$$

51. $a_1 = 28$ and $a_{k+1} = a_k - 4$

$a_1 = 28$

$a_2 = a_1 - 4 = 28 - 4 = 24$

$a_3 = a_2 - 4 = 24 - 4 = 20$

$a_4 = a_3 - 4 = 20 - 4 = 16$

$a_5 = a_4 - 4 = 16 - 4 = 12$

52. $a_1 = 15, \quad a_{k+1} = a_k + 3$

$a_1 = 15$

$a_2 = a_1 + 3 = 15 + 3 = 18$

$a_3 = a_2 + 3 = 18 + 3 = 21$

$a_4 = a_3 + 3 = 21 + 3 = 24$

$a_5 = a_4 + 3 = 24 + 3 = 27$

53. $a_1 = 3$ and $a_{k+1} = 2(a_k - 1)$

$a_1 = 3$

$a_2 = 2(a_1 - 1) = 2(3 - 1) = 4$

$a_3 = 2(a_2 - 1) = 2(4 - 1) = 6$

$a_4 = 2(a_3 - 1) = 2(6 - 1) = 10$

$a_5 = 2(a_4 - 1) = 2(10 - 1) = 18$

54. $a_1 = 32, \quad a_{k+1} = \frac{1}{2}a_k$

$a_1 = 32$

$a_2 = \frac{1}{2}a_1 = \frac{1}{2}(32) = 16$

$a_3 = \frac{1}{2}a_2 = \frac{1}{2}(16) = 8$

$a_4 = \frac{1}{2}a_3 = \frac{1}{2}(8) = 4$

$a_5 = \frac{1}{2}a_4 = \frac{1}{2}(4) = 2$

55. $a_1 = 6$ and $a_{k+1} = a_k + 2$

$a_1 = 6$

$a_2 = a_1 + 2 = 6 + 2 = 8$

$a_3 = a_2 + 2 = 8 + 2 = 10$

$a_4 = a_3 + 2 = 10 + 2 = 12$

$a_5 = a_4 + 2 = 12 + 2 = 14$

In general, $a_n = 2n + 4$.

56. $a_1 = 25, \quad a_{k+1} = a_k - 5$

$a_1 = 25$

$a_2 = a_1 - 5 = 25 - 5 = 20$

$a_3 = a_2 - 5 = 20 - 5 = 15$

$a_4 = a_3 - 5 = 15 - 5 = 10$

$a_5 = a_4 - 5 = 10 - 5 = 5$

In general, $a_n = 30 - 5n$.

57. $a_1 = 81$ and $a_{k+1} = \frac{1}{3}a_k$

$a_1 = 81$

$a_2 = \frac{1}{3}a_1 = \frac{1}{3}(81) = 27$

$a_3 = \frac{1}{3}a_2 = \frac{1}{3}(27) = 9$

$a_4 = \frac{1}{3}a_3 = \frac{1}{3}(9) = 3$

$a_5 = \frac{1}{3}a_4 = \frac{1}{3}(3) = 1$

In general,

$$a_n = 81\left(\frac{1}{3}\right)^{n-1} = 81(3)\left(\frac{1}{3}\right)^n = \frac{243}{3^n}.$$

58. $a_1 = 14, \quad a_{k+1} = (-2)a_k$

$a_1 = 14$

$a_2 = (-2)a_1 = (-2)(14) = -28$

$a_3 = (-2)a_2 = (-2)(-28) = 56$

$a_4 = (-2)a_3 = (-2)(56) = -112$

$a_5 = (-2)(a_4) = (-2)(-112) = 224$

In general, $a_n = 14(-2)^{n-1}$.

59. $a_n = \frac{3^n}{n!}$

$a_0 = \frac{3^0}{0!} = 1$

$a_1 = \frac{3^1}{1!} = 3$

$a_2 = \frac{3^2}{2!} = \frac{9}{2}$

$a_3 = \frac{3^3}{3!} = \frac{27}{6} = \frac{9}{2}$

$a_4 = \frac{3^4}{4!} = \frac{81}{24} = \frac{27}{8}$

60. $a_n = \dfrac{n!}{n}$

$a_0 = \dfrac{0!}{0} = \text{undefined}$

$a_1 = \dfrac{1!}{1} = \dfrac{1}{1} = 1$

$a_2 = \dfrac{2!}{2} = \dfrac{2 \cdot 1}{2} = 1$

$a_3 = \dfrac{3!}{3} = \dfrac{3 \cdot 2 \cdot 1}{3} = 2$

$a_4 = \dfrac{4!}{4} = \dfrac{4 \cdot 3 \cdot 2 \cdot 1}{4} = 6$

61. $a_n = \dfrac{1}{(n+1)!}$

$a_0 = \dfrac{1}{1!} = 1$

$a_1 = \dfrac{1}{2!} = \dfrac{1}{2}$

$a_2 = \dfrac{1}{3!} = \dfrac{1}{6}$

$a_3 = \dfrac{1}{4!} = \dfrac{1}{24}$

$a_4 = \dfrac{1}{5!} = \dfrac{1}{120}$

62. $a_n = \dfrac{n^2}{(n+1)!}$

$a_0 = \dfrac{0^2}{(0+1)!} = \dfrac{0}{1} = 0$

$a_1 = \dfrac{1^2}{(1+1)!} = \dfrac{1}{2 \cdot 1} = \dfrac{1}{2}$

$a_2 = \dfrac{2^2}{(2+1)!} = \dfrac{4}{3 \cdot 2 \cdot 1} = \dfrac{2}{3}$

$a_3 = \dfrac{3^2}{(3+1)!} = \dfrac{9}{4 \cdot 3 \cdot 2 \cdot 1} = \dfrac{3}{8}$

$a_4 = \dfrac{4^2}{(4+1)!} = \dfrac{16}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \dfrac{2}{15}$

63. $a_n = \dfrac{(-1)^{2n}}{(2n)!} = \dfrac{1}{(2n)!}$

$a_0 = \dfrac{1}{0!} = 1$

$a_1 = \dfrac{1}{2!} = \dfrac{1}{2}$

$a_2 = \dfrac{1}{4!} = \dfrac{1}{24}$

$a_3 = \dfrac{1}{6!} = \dfrac{1}{720}$

$a_4 = \dfrac{1}{8!} = \dfrac{1}{40{,}320}$

64. $a_n = \dfrac{(-1)^{2n+1}}{(2n+1)!}$

$a_0 = \dfrac{(-1)^{2(0)+1}}{(2 \cdot 0 + 1)!} = \dfrac{(-1)^1}{1!} = \dfrac{-1}{1} = -1$

$a_1 = \dfrac{(-1)^{2 \cdot 1 + 1}}{(2 \cdot 1 + 1)!} = \dfrac{(-1)^3}{3!} = \dfrac{-1}{6} = -\dfrac{1}{6}$

$a_2 = \dfrac{(-1)^{2 \cdot 2 + 1}}{(2 \cdot 2 + 1)!} = \dfrac{(-1)^5}{5!} = \dfrac{-1}{120} = -\dfrac{1}{120}$

$a_3 = \dfrac{(-1)^{2 \cdot 3 + 1}}{(2 \cdot 3 + 1)!} = \dfrac{(-1)^7}{7!} = \dfrac{-1}{5040} = -\dfrac{1}{5040}$

$a_4 = \dfrac{(-1)^{2 \cdot 4 + 1}}{(2 \cdot 4 + 1)!} = \dfrac{(-1)^9}{9!} = \dfrac{-1}{362{,}880} = -\dfrac{1}{362{,}880}$

65. $\dfrac{4!}{6!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot 4}}{\cancel{1 \cdot 2 \cdot 3 \cdot 4} \cdot 5 \cdot 6} = \dfrac{1}{5 \cdot 6} = \dfrac{1}{30}$

66. $\dfrac{5!}{8!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}}{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot 6 \cdot 7 \cdot 8} = \dfrac{1}{6 \cdot 7 \cdot 8} = \dfrac{1}{336}$

67. $\dfrac{10!}{8!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \cdot 9 \cdot 10}{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}} = \dfrac{9 \cdot 10}{1} = 90$

68. $\dfrac{25!}{23!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 23} \cdot 24 \cdot 25}{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 23}} = \dfrac{24 \cdot 25}{1} = 600$

69. $\dfrac{(n+1)!}{n!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \cdot (n+1)}{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}} = \dfrac{n+1}{1}$

$= n + 1$

70. $\dfrac{(n+2)!}{n!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \cdot (n+1) \cdot (n+2)}{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}}$

$= (n+1)(n+2)$

71. $\dfrac{(2n-1)!}{(2n+1)!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n-1)}}{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n-1)} \cdot (2n) \cdot (2n+1)}$

$= \dfrac{1}{2n(2n+1)}$

72. $\dfrac{(3n+1)!}{(3n)!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (3n)} \cdot (3n+1)}{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (3n)}}$

$= \dfrac{3n+1}{1} = 3n + 1$

73. $\displaystyle\sum_{i=1}^{5} (2i+1) = (2+1) + (4+1) + (6+1) + (8+1) + (10+1) = 35$

74. $\displaystyle\sum_{i=1}^{6} (3i-1) = (3 \cdot 1 - 1) + (3 \cdot 2 - 1) + (3 \cdot 3 - 1) + (3 \cdot 4 - 1) + (3 \cdot 5 - 1) + (3 \cdot 6 - 1) = 57$

75. $\displaystyle\sum_{k=1}^{4} 10 = 10 + 10 + 10 + 10 = 40$

76. $\displaystyle\sum_{k=1}^{5} 5 = 5 + 5 + 5 + 5 + 5 = 25$

77. $\sum_{i=0}^{4} i^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30$

78. $\sum_{i=0}^{5} 2i^2 = 2(0^2) + 2(1^2) + 2(2^2) + 2(3^2) + 2(4^2) + 2(5^2)$
$= 110$

79. $\sum_{k=0}^{3} \frac{1}{k^2+1} = \frac{1}{1} + \frac{1}{1+1} + \frac{1}{4+1} + \frac{1}{9+1} = \frac{9}{5}$

80. $\sum_{j=3}^{5} \frac{1}{j^2-3} = \frac{1}{3^2-3} + \frac{1}{4^2-3} + \frac{1}{5^2-3} = \frac{124}{429}$

81. $\sum_{k=2}^{5} (k+1)^2(k-3) = (3)^2(-1) + (4)^2(0) + (5)^2(1) + (6)^2(2) = 88$

82. $\sum_{i=1}^{4} [(i-1)^2 + (i+1)^3] = [(0)^2 + (2)^3] + [(1)^2 + (3)^3] + [(2)^2 + (4)^3] + [(3)^2 + (5)^3] = 238$

83. $\sum_{i=1}^{4} 2^i = 2^1 + 2^2 + 2^3 + 2^4 = 30$

84. $\sum_{j=0}^{4} (-2)^j = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4$
$= 11$

85. $\sum_{j=1}^{6} (24-3j) = 81$

86. $\sum_{j=1}^{10} \frac{3}{j+1} \approx 6.06$

87. $\sum_{k=0}^{4} \frac{(-1)^k}{k+1} = \frac{47}{60}$

88. $\sum_{k=0}^{4} \frac{(-1)^k}{k!} = \frac{3}{8}$

89. $\frac{1}{3(1)} + \frac{1}{3(2)} + \frac{1}{3(3)} + \cdots + \frac{1}{3(9)} = \sum_{i=1}^{9} \frac{1}{3i}$

90. $\frac{5}{1+1} + \frac{5}{1+2} + \frac{5}{1+3} + \cdots + \frac{5}{1+15} = \sum_{i=1}^{15} \frac{5}{1+i}$

91. $\left[2\left(\frac{1}{8}\right) + 3\right] + \left[2\left(\frac{2}{8}\right) + 3\right] + \left[2\left(\frac{3}{8}\right) + 3\right] + \cdots + \left[2\left(\frac{8}{8}\right) + 3\right] = \sum_{i=1}^{8} \left[2\left(\frac{i}{8}\right) + 3\right]$

92. $\left[1 - \left(\frac{1}{6}\right)^2\right] + \left[1 - \left(\frac{2}{6}\right)^2\right] + \cdots + \left[1 - \left(\frac{6}{6}\right)^2\right] = \sum_{k=1}^{6} \left[1 - \left(\frac{k}{6}\right)^2\right]$

93. $3 - 9 + 27 - 81 + 243 - 729 = \sum_{i=1}^{6} (-1)^{i+1} 3^i$

94. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{128} = \frac{1}{2^0} - \frac{1}{2^1} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots - \frac{1}{2^7} = \sum_{n=0}^{7} \left(-\frac{1}{2}\right)^n$

95. $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots - \frac{1}{20^2} = \sum_{i=1}^{20} \frac{(-1)^{i+1}}{i^2}$

96. $\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{10 \cdot 12} = \sum_{k=1}^{10} \frac{1}{k(k+2)}$

97. $\frac{1}{4} + \frac{3}{8} + \frac{7}{16} + \frac{15}{32} + \frac{31}{64} = \sum_{i=1}^{5} \frac{2^i - 1}{2^{i+1}}$

98. $\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} + \frac{120}{32} + \frac{720}{64} = \sum_{k=1}^{6} \frac{k!}{2^k}$

99. $\sum_{i=1}^{4} 5\left(\frac{1}{2}\right)^i = 5\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^4 = \frac{75}{16}$

100. $\sum_{i=1}^{5} 2\left(\frac{1}{3}\right)^i = 2\left(\frac{1}{3}\right)^1 + 2\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^4 + 2\left(\frac{1}{3}\right)^5$
$= \frac{242}{243}$

101. $\sum_{n=1}^{3} 4\left(-\frac{1}{2}\right)^n = 4\left(-\frac{1}{2}\right) + 4\left(-\frac{1}{2}\right)^2 + 4\left(-\frac{1}{2}\right)^3 = -\frac{3}{2}$

102. $\sum_{n=1}^{4} 8\left(-\frac{1}{4}\right)^n = 8\left(-\frac{1}{4}\right)^1 + 8\left(-\frac{1}{4}\right)^2 + 8\left(-\frac{1}{4}\right)^3 + 8\left(-\frac{1}{4}\right)^4$
$= -\frac{51}{32}$

103. $\sum_{i=1}^{\infty} 6\left(\frac{1}{10}\right)^i = 0.6 + 0.06 + 0.006 + 0.0006 + \cdots = \frac{2}{3}$

104. $$\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^k = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \frac{1}{10^5} + \cdots$$
$$= 0.1 + 0.01 + 0.001 + 0.0001 + 0.00001 + \cdots$$
$$= 0.11111\ldots$$
$$= \frac{1}{9}$$

105. By using a calculator, we have

$$\sum_{k=1}^{10} 7\left(\frac{1}{10}\right)^k \approx 0.7777777777$$
$$\sum_{k=1}^{50} 7\left(\frac{1}{10}\right)^k \approx 0.7777777778$$
$$\sum_{k=1}^{100} 7\left(\frac{1}{10}\right)^k \approx \frac{7}{9}.$$

The terms approach zero as $n \longrightarrow \infty$.

Thus, we conclude that $\sum_{k=1}^{\infty} 7\left(\frac{1}{10}\right)^k = \frac{7}{9}$.

106. $$\sum_{i=1}^{\infty} 2\left(\frac{1}{10}\right)^i = 2\left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots\right)$$
$$= 2(0.1 + 0.01 + 0.001 + 0.0001 + \cdots)$$
$$= 2(0.111\ldots)$$
$$= 0.222\ldots$$
$$= \frac{2}{9}$$

107. $A_n = 5000\left(1 + \frac{0.08}{4}\right)^n,\ n = 1, 2, 3, \ldots$

(a) $A_1 = \$5100.00$

$A_2 = \$5202.00$

$A_3 = \$5306.04$

$A_4 = \$5412.16$

$A_5 = \$5520.40$

$A_6 = \$5630.81$

$A_7 = \$5743.43$

$A_8 = \$5858.30$

(b) $A_{40} = \$11{,}040.20$

108. (a) $A_1 = 100(101)[(1.01)^1 - 1] = \$101.00$

$A_2 = 100(101)[(1.01)^2 - 1] = \$203.01$

$A_3 = 100(101)[(1.01)^3 - 1] \approx \$306.04$

$A_4 = 100(101)[(1.01)^4 - 1] \approx \$410.10$

$A_5 = 100(101)[(1.01)^5 - 1] \approx \$515.20$

$A_6 = 100(101)[(1.01)^6 - 1] \approx \$621.35$

(b) $A_{60} = 100(101)[(1.01)^{60} - 1] \approx \$8248.64$

(c) $A_{240} = 100(101)[(1.01)^{240} - 1] \approx \$99{,}914.79$

109. (a) Linear model: $a_n \approx 60.57n - 182$

(b) Quadratic model: $a_n \approx 1.61n^2 + 26.8n - 9.5$

(c)

Year	n	Actual Data	Linear Model	Quadratic Model
1998	8	311	303	308
1999	9	357	363	362
2000	10	419	424	420
2001	11	481	484	480
2002	12	548	545	544
2003	13	608	605	611

The quadratic model is a better fit.

(d) For the year 2008 we have the following predictions:

Linear model: 908 stores

Quadratic model: 995 stores

Since the quadratic model is a better fit, the predicted number of stores in 2008 is 995.

110. (a) $a_n = 0.0457n^3 - 0.3498n^2 - 9.04n + 121.3, n = 5, \ldots, 13.$

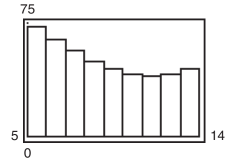
$a_5 = 73.1$ $\quad a_{10} = 41.6$

$a_6 = 64.3$ $\quad a_{11} = 40.4$

$a_7 = 56.6$ $\quad a_{12} = 41.4$

$a_8 = 50.0$ $\quad a_{13} = 45.1$

$a_9 = 44.9$



(b) The number of cases reported fluctuates.

111. (a) $a_n = 2.7698n^3 - 61.372n^2 + 600.00n + 3102.9$

$a_0 = \$3102.9$ billion $\quad a_7 \approx \$5245.7$ billion

$a_1 \approx \$3644.3$ billion $\quad a_8 \approx \$5393.2$ billion

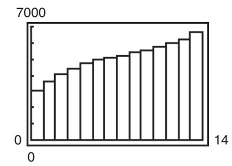
$a_2 \approx \$4079.6$ billion $\quad a_9 \approx \$5551.0$ billion

$a_3 \approx \$4425.3$ billion $\quad a_{10} = \$5735.5$ billion

$a_4 \approx \$4698.2$ billion $\quad a_{11} \approx \$5963.5$ billion

$a_5 \approx \$4914.8$ billion $\quad a_{12} \approx \$6251.5$ billion

$a_6 \approx \$5091.8$ billion $\quad a_{13} \approx \$6616.3$ billion



(b) The federal debt is increasing.

112. $\sum_{n=6}^{13}(46.609n^2 - 119.84n - 1125.8) = \$17{,}495$ million

The results from the model and the figure (which are approximations) are very similar.

113. True, $\sum_{i=1}^{4}(i^2 + 2i) = \sum_{i=1}^{4} i^2 + 2\sum_{i=1}^{4} i$ by the Properties of Sums.

114. $\sum_{j=1}^{4} 2^j = \sum_{j=3}^{6} 2^{j-2}$

True, because $2^1 + 2^2 + 2^3 + 2^4 = 2^{3-2} + 2^{4-2} + 2^{5-2} + 2^{6-2}$.

115. $a_1 = 1, a_2 = 1, \; a_{k+2} = a_{k+1} + a_k, k \geq 1$

$a_1 = 1$

$a_2 = 1$

$a_3 = 1 + 1 = 2$

$a_4 = 2 + 1 = 3$

$a_5 = 3 + 2 = 5$

$a_6 = 5 + 3 = 8$

$a_7 = 8 + 5 = 13$

$a_8 = 13 + 8 = 21$

$a_9 = 21 + 13 = 34$

$a_{10} = 34 + 21 = 55$

$a_{11} = 55 + 34 = 89$

$a_{12} = 89 + 55 = 144$

$b_1 = \frac{1}{1} = 1$

$b_2 = \frac{2}{1} = 2$

$b_3 = \frac{3}{2}$

$b_4 = \frac{5}{3}$

$b_5 = \frac{8}{5}$

$b_6 = \frac{13}{8}$

$b_7 = \frac{21}{13}$

$b_8 = \frac{34}{21}$

$b_9 = \frac{55}{34}$

$b_{10} = \frac{89}{55}$

116. $b_n = \frac{a_{n+1}}{a_n}; b_1 = 1, b_2 = 2, b_3 = \frac{3}{2}, b_4 = \frac{5}{3}, \ldots$

$b_2 = 1 + \frac{1}{b_1} = 1 + \frac{1}{1} = 2$

$b_3 = 1 + \frac{1}{b_2} = 1 + \frac{1}{2} = \frac{3}{2}$

$b_4 = 1 + \frac{1}{b_3} = 1 + \frac{2}{3} = \frac{5}{3}$

$b_5 = 1 + \frac{1}{b_4} = 1 + \frac{3}{5} = \frac{8}{5}$

$b_n = 1 + \frac{1}{b_{n-1}}$

117. $\dfrac{327.15 + 785.69 + 433.04 + 265.38 + 604.12 + 590.30}{6} \approx \$500.95$

118. $\bar{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i = \dfrac{1.899 + 1.959 + 1.919 + 1.939 + 1.999}{5}$

$= \$1.943$

119. $\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x}$

$= \left(\sum_{i=1}^{n} x_i\right) - n\bar{x}$

$= \left(\sum_{i=1}^{n} x_i\right) - n\left(\dfrac{1}{n}\sum_{i=1}^{n} x_i\right)$

$= 0$

120. $\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2 = \sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right) = \sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + n\bar{x}^2$

$= \sum_{i=1}^{n} x_i^2 - 2 \cdot \dfrac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i + n \cdot \dfrac{1}{n}\sum_{i=1}^{n} x_i \cdot \dfrac{1}{n}\sum_{i=1}^{n} x_i$

$= \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i\left(-\dfrac{2}{n} + \dfrac{1}{n}\right) = \sum_{i=1}^{n} x_i^2 - \dfrac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2$

121. $a_n = \dfrac{x^n}{n!}$

$a_1 = \dfrac{x^1}{1!} = x$

$a_2 = \dfrac{x^2}{2!} = \dfrac{x^2}{2}$

$a_3 = \dfrac{x^3}{3!} = \dfrac{x^3}{6}$

$a_4 = \dfrac{x^4}{4!} = \dfrac{x^4}{24}$

$a_5 = \dfrac{x^5}{5!} = \dfrac{x^5}{120}$

122. $a_n = \dfrac{(-1)^n x^{2n+1}}{2n + 1}$

$a_1 = \dfrac{(-1)^1 x^{2(1)+1}}{2(1) + 1} = -\dfrac{x^3}{3}$

$a_2 = \dfrac{(-1)^2 x^{2(2)+1}}{2(2) + 1} = \dfrac{x^5}{5}$

$a_3 = \dfrac{(-1)^3 x^{2(3)+1}}{2(3) + 1} = -\dfrac{x^7}{7}$

$a_4 = \dfrac{(-1)^4 x^{2(4)+1}}{2(4) + 1} = \dfrac{x^9}{9}$

$a_5 = \dfrac{(-1)^5 x^{2(5)+1}}{2(5) + 1} = -\dfrac{x^{11}}{11}$

123. $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$

$a_1 = \dfrac{-x^2}{2!} = -\dfrac{x^2}{2}$

$a_2 = \dfrac{x^4}{4!} = \dfrac{x^4}{24}$

$a_3 = \dfrac{-x^6}{6!} = -\dfrac{x^6}{720}$

$a_4 = \dfrac{x^8}{8!} = \dfrac{x^8}{40{,}320}$

$a_5 = \dfrac{-x^{10}}{10!} = -\dfrac{x^{10}}{3{,}628{,}800}$

124. $a_n = \dfrac{(-1)^n x^{2n+1}}{(2n + 1)!}$

$a_1 = \dfrac{(-1)^1 x^{2(1)+1}}{(2(1) + 1)!} = -\dfrac{x^3}{3!} = -\dfrac{x^3}{6}$

$a_2 = \dfrac{(-1)^2 x^{2(2)+1}}{(2(2) + 1)!} = \dfrac{x^5}{5!} = \dfrac{x^5}{120}$

$a_3 = \dfrac{(-1)^3 x^{2(3)+1}}{(2(3) + 1)!} = -\dfrac{x^7}{7!} = -\dfrac{x^7}{5040}$

$a_4 = \dfrac{(-1)^4 x^{2(4)+1}}{(2(4) + 1)!} = \dfrac{x^9}{9!} = \dfrac{x^9}{362{,}880}$

$a_5 = \dfrac{(-1)^5 x^{2(5)+1}}{(2(5) + 1)!} = -\dfrac{x^{11}}{11!} = -\dfrac{x^{11}}{39{,}916{,}800}$

125. $f(x) = 4x - 3$ is one-to-one, so it has an inverse.

$y = 4x - 3$

$x = 4y - 3$

$\dfrac{x + 3}{4} = y$

$f^{-1}(x) = \dfrac{x + 3}{4}$

126. $g(x) = \dfrac{3}{x}$

$$y = \frac{3}{x}$$

$$x = \frac{3}{y}$$

$$xy = 3$$

$$y = \frac{3}{x}$$

This is a function of x, so f has in inverse.

$f^{-1}(x) = \dfrac{3}{x}, x \neq 0$

127. $h(x) = \sqrt{5x + 1}$ is one-to-one, so it has an inverse.

Domain: $x \geq -\dfrac{1}{5}$

Range: $y \geq 0$

$y = \sqrt{5x + 1}, x \geq -\dfrac{1}{5}, y \geq 0$

$x = \sqrt{5y + 1}, x \geq 0, y \geq -\dfrac{1}{5}$

$x^2 = 5y + 1, x \geq 0$

$\dfrac{x^2 - 1}{5} = y, x \geq 0$

$h^{-1}(x) = \dfrac{x^2 - 1}{5} = \dfrac{1}{5}(x^2 - 1), x \geq 0$

128. $f(x) = (x - 1)^2$

$$y = (x - 1)^2$$

$$x = (y - 1)^2$$

$$\pm\sqrt{x} = y - 1$$

$$1 \pm \sqrt{x} = y$$

This does not represent y as a function of x, so f does not have an inverse.

129. (a) $A - B = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 6 & -3 \end{bmatrix} = \begin{bmatrix} 6 - (-2) & 5 - 4 \\ 3 - 6 & 4 - (-3) \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ -3 & 7 \end{bmatrix}$

(b) $4B - 3A = 4\begin{bmatrix} -2 & 4 \\ 6 & -3 \end{bmatrix} - 3\begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -8 - 18 & 16 - 15 \\ 24 - 9 & -12 - 12 \end{bmatrix} = \begin{bmatrix} -26 & 1 \\ 15 & -24 \end{bmatrix}$

(c) $AB = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 6 & -3 \end{bmatrix} = \begin{bmatrix} -12 + 30 & 24 - 15 \\ -6 + 24 & 12 - 12 \end{bmatrix} = \begin{bmatrix} 18 & 9 \\ 18 & 0 \end{bmatrix}$

(d) $BA = \begin{bmatrix} -2 & 4 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -12 + 16 & -10 + 12 \\ 36 - 9 & 30 - 12 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 27 & 18 \end{bmatrix}$

130. (a) $A - B = \begin{bmatrix} 10 & 7 \\ -4 & 6 \end{bmatrix} - \begin{bmatrix} 0 & -12 \\ 8 & 11 \end{bmatrix} = \begin{bmatrix} 10 - 0 & 7 - (-12) \\ -4 - 8 & 6 - 11 \end{bmatrix} = \begin{bmatrix} 10 & 19 \\ -12 & -5 \end{bmatrix}$

(b) $4B - 3A = 4\begin{bmatrix} 0 & -12 \\ 8 & 11 \end{bmatrix} - 3\begin{bmatrix} 10 & 7 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 0 - 30 & -48 - 21 \\ 32 + 12 & 44 - 18 \end{bmatrix} = \begin{bmatrix} -30 & -69 \\ 44 & 26 \end{bmatrix}$

(c) $AB = \begin{bmatrix} 10 & 7 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 0 & -12 \\ 8 & 11 \end{bmatrix} = \begin{bmatrix} 0 + 56 & -120 + 77 \\ 0 + 48 & 48 + 66 \end{bmatrix} = \begin{bmatrix} 56 & -43 \\ 48 & 114 \end{bmatrix}$

(d) $BA = \begin{bmatrix} 0 & -12 \\ 8 & 11 \end{bmatrix}\begin{bmatrix} 10 & 7 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 0 + 48 & 0 - 72 \\ 80 - 44 & 56 + 66 \end{bmatrix} = \begin{bmatrix} 48 & -72 \\ 36 & 122 \end{bmatrix}$

131. (a) $A - B = \begin{bmatrix} -2 & -3 & 6 \\ 4 & 5 & 7 \\ 1 & 7 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 6 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -2-1 & -3-4 & 6-2 \\ 4-0 & 5-1 & 7-6 \\ 1-0 & 7-3 & 4-1 \end{bmatrix} = \begin{bmatrix} -3 & -7 & 4 \\ 4 & 4 & 1 \\ 1 & 4 & 3 \end{bmatrix}$

(b) $4B - 3A = 4\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 6 \\ 0 & 3 & 1 \end{bmatrix} - 3\begin{bmatrix} -2 & -3 & 6 \\ 4 & 5 & 7 \\ 1 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 4-(-6) & 16-(-9) & 8-18 \\ 0-12 & 4-15 & 24-21 \\ 0-3 & 12-21 & 4-12 \end{bmatrix} = \begin{bmatrix} 10 & 25 & -10 \\ -12 & -11 & 3 \\ -3 & -9 & -8 \end{bmatrix}$

(c) $AB = \begin{bmatrix} -2 & -3 & 6 \\ 4 & 5 & 7 \\ 1 & 7 & 4 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 6 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -2+0+0 & -8-3+18 & -4-18+6 \\ 4+0+0 & 16+5+21 & 8+30+7 \\ 1+0+0 & 4+7+12 & 2+42+4 \end{bmatrix} = \begin{bmatrix} -2 & 7 & -16 \\ 4 & 42 & 45 \\ 1 & 23 & 48 \end{bmatrix}$

(d) $BA = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 6 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} -2 & -3 & 6 \\ 4 & 5 & 7 \\ 1 & 7 & 4 \end{bmatrix} = \begin{bmatrix} -2+16+2 & -3+20+14 & 6+28+8 \\ 0+4+6 & 0+5+42 & 0+7+24 \\ 0+12+1 & 0+15+7 & 0+21+4 \end{bmatrix} = \begin{bmatrix} 16 & 31 & 42 \\ 10 & 47 & 31 \\ 13 & 22 & 25 \end{bmatrix}$

132. (a) $A - B = \begin{bmatrix} -1 & 4 & 0 \\ 5 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 4 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1-0 & 4-4 & 0-0 \\ 5-3 & 1-1 & 2-(-2) \\ 0-(-1) & -1-0 & 3-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 4 \\ 1 & -1 & 1 \end{bmatrix}$

(b) $4B - 3A = 4\begin{bmatrix} 0 & 4 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix} - 3\begin{bmatrix} -1 & 4 & 0 \\ 5 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0-(-3) & 16-12 & 0-0 \\ 12-15 & 4-3 & -8-6 \\ -4-0 & 0-(-3) & 8-9 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 0 \\ -3 & 1 & -14 \\ -4 & 3 & -1 \end{bmatrix}$

(c) $AB = \begin{bmatrix} -1 & 4 & 0 \\ 5 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 4 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0+12+0 & -4+4+0 & 0-8+0 \\ 0+3-2 & 20+1+0 & 0-2+4 \\ 0-3-3 & 0-1+0 & 0+2+6 \end{bmatrix} = \begin{bmatrix} 12 & 0 & -8 \\ 1 & 21 & 2 \\ -6 & -1 & 8 \end{bmatrix}$

(d) $BA = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 4 & 0 \\ 5 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0+20+0 & 0+4+0 & 0+8+0 \\ -3+5+0 & 12+1+2 & 0+2-6 \\ 1+0+0 & -4+0-2 & 0+0+6 \end{bmatrix} = \begin{bmatrix} 20 & 4 & 8 \\ 2 & 15 & -4 \\ 1 & -6 & 6 \end{bmatrix}$

133. $|A| = \begin{vmatrix} 3 & 5 \\ -1 & 7 \end{vmatrix} = 3(7) - 5(-1) = 26$

134. $\begin{vmatrix} -2 & 8 \\ 12 & 15 \end{vmatrix} = -2(15) - 8(12) = -126$

135. $|A| = \begin{vmatrix} 3 & 4 & 5 \\ 0 & 7 & 3 \\ 4 & 9 & -1 \end{vmatrix} = 3\begin{vmatrix} 7 & 3 \\ 9 & -1 \end{vmatrix} + 4\begin{vmatrix} 4 & 5 \\ 7 & 3 \end{vmatrix}$

$= 3[7(-1) - 3(9)] + 4[4(3) - 5(7)] = -194$

136. $|A| = 16(C_{11}) + 9(C_{21}) - 2(C_{31}) - 4(C_{41})$

$C_{11} = (-1)^{1+1}\begin{vmatrix} 8 & 3 & 7 \\ -1 & 12 & 3 \\ 6 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 8 & 3 & 7 \\ -1 & 12 & 3 \\ 6 & 2 & 1 \end{vmatrix}$

$= 8\begin{vmatrix} 12 & 3 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ 6 & 1 \end{vmatrix} + 7\begin{vmatrix} -1 & 12 \\ 6 & 2 \end{vmatrix}$

$= 8(12 - 6) - 3(-1 - 18) + 7(-2 - 72) = -413$

$C_{21} = (-1)^{2+1}\begin{vmatrix} 11 & 10 & 2 \\ -1 & 12 & 3 \\ 6 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -11 & -10 & -2 \\ 1 & -12 & -3 \\ -6 & -2 & -1 \end{vmatrix}$

$= -11\begin{vmatrix} -12 & -3 \\ -2 & -1 \end{vmatrix} - 1\begin{vmatrix} -10 & -2 \\ -2 & -1 \end{vmatrix} - 6\begin{vmatrix} -10 & -2 \\ -12 & -3 \end{vmatrix}$

$= -11(12 - 6) - 1(10 - 4) - 6(30 - 24) = -108$

—CONTINUED—

136. —CONTINUED—

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 11 & 10 & 2 \\ 8 & 3 & 7 \\ 6 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 11 & 10 & 2 \\ 8 & 3 & 7 \\ 6 & 2 & 1 \end{vmatrix}$$

$$= 11\begin{vmatrix} 3 & 7 \\ 2 & 1 \end{vmatrix} - 8\begin{vmatrix} 10 & 2 \\ 2 & 1 \end{vmatrix} + 6\begin{vmatrix} 10 & 2 \\ 3 & 7 \end{vmatrix}$$

$$= 11(3 - 14) - 8(10 - 4) + 6(70 - 6) = 215$$

$$C_{41} = (-1)^{4+1}\begin{vmatrix} 11 & 10 & 2 \\ 8 & 3 & 7 \\ -1 & 12 & 3 \end{vmatrix} = \begin{vmatrix} -11 & -10 & -2 \\ -8 & -3 & -7 \\ 1 & -12 & -3 \end{vmatrix}$$

$$= -11\begin{vmatrix} -3 & -7 \\ -12 & -3 \end{vmatrix} - (-8)\begin{vmatrix} -10 & -2 \\ -12 & -3 \end{vmatrix} + 1\begin{vmatrix} -10 & -2 \\ -3 & -7 \end{vmatrix}$$

$$= -11(9 - 84) + 8(30 - 24) + 1(70 - 6) = 937$$

So, $|A| = 16(-413) + 9(-108) - 2(215) - 4(937)$

$= -11{,}758.$

Section 9.2 Arithmetic Sequences and Partial Sums

- You should be able to recognize an arithmetic sequence, find its common difference, and find its nth term.
- You should be able to find the nth partial sum of an arithmetic sequence by using the formula

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Vocabulary Check

1. arithmetic; common
2. $a_n = dn + c$
3. sum of a finite arithmetic sequence

1. 10, 8, 6, 4, 2, . . .

Arithmetic sequence, $d = -2$

2. 4, 7, 10, 13, 16, . . .

Arithmetic sequence, $d = 3$

3. 1, 2, 4, 8, 16, . . .

Not an arithmetic sequence

4. 80, 40, 20, 10, 5, . . .

Not an arithmetic sequence

5. $\frac{9}{4}, 2, \frac{7}{4}, \frac{3}{2}, \frac{5}{4}, \ldots$

Arithmetic sequence, $d = -\frac{1}{4}$

6. $3, \frac{5}{2}, 2, \frac{3}{2}, 1, \ldots$

Arithmetic sequence, $d = -\frac{1}{2}$

7. $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{6}, \ldots$

Not an arithmetic sequence

8. 5.3, 5.7, 6.1, 6.5, 6.9, . . .

Arithmetic sequence, $d = 0.4$

9. ln 1, ln 2, ln 3, ln 4, ln 5, . . .

Not an arithmetic sequence

10. $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

Not an arithmetic sequence

11. $a_n = 5 + 3n$

8, 11, 14, 17, 20

Arithmetic sequence, $d = 3$

12. $a_n = 100 - 3n$

97, 94, 91, 88, 85

Arithmetic sequence, $d = -3$

13. $a_n = 3 - 4(n - 2)$

$7, 3, -1, -5, -9$

Arithmetic sequence, $d = -4$

14. $a_n = 1 + (n - 1)4$

$1, 5, 9, 13, 17$

Arithmetic sequence, $d = 4$

15. $a_n = (-1)^n$

$-1, 1, -1, 1, -1$

Not an arithmetic sequence

16. $a_n = 2^{n-1}$

$1, 2, 4, 8, 16$

Not an arithmetic sequence

17. $a_n = \dfrac{(-1)^n 3}{n}$

$-3, \dfrac{3}{2}, -1, \dfrac{3}{4}, -\dfrac{3}{5}$

Not an arithmetic sequence

18. $a_n = (2^n)n$

$2, 8, 24, 64, 160$

Not an arithmetic sequence

19. $a_1 = 1,\ d = 3$

$a_n = a_1 + (n - 1)d = 1 + (n - 1)(3) = 3n - 2$

20. $a_1 = 15, d = 4$

$a_n = a_1 + (n - 1)d = 15 + (n - 1)4$

$= 4n + 11$

21. $a_1 = 100,\ d = -8$

$a_n = a_1 + (n - 1)d = 100 + (n - 1)(-8)$

$= -8n + 108$

22. $a_1 = 0, d = -\frac{2}{3}$

$a_n = a_1 + (n - 1)d = (n - 1)\left(-\frac{2}{3}\right)$

$= -\frac{2}{3}n + \frac{2}{3}$

23. $a_1 = x,\ d = 2x$

$a_n = a_1 + (n - 1)d = x + (n - 1)(2x) = 2xn - x$

24. $a_1 = -y, d = 5y$

$a_n = a_1 + (n - 1)d = -y + (n - 1)(5y)$

$= 5yn - 6y$

25. $4, \frac{3}{2}, -1, -\frac{7}{2}, \ldots$

$d = -\frac{5}{2}$

$a_n = a_1 + (n - 1)d = 4 + (n - 1)\left(-\frac{5}{2}\right) = -\frac{5}{2}n + \frac{13}{2}$

26. $10, 5, 0, -5, -10, \ldots$

$d = -5$

$a_n = a_1 + (n - 1)d = 10 + (n - 1)(-5) = -5n + 15$

27. $a_1 = 5,\ a_4 = 15$

$a_4 = a_1 + 3d \implies 15 = 5 + 3d \implies d = \frac{10}{3}$

$a_n = a_1 + (n - 1)d = 5 + (n - 1)\left(\frac{10}{3}\right) = \frac{10}{3}n + \frac{5}{3}$

28. $a_1 = -4, a_5 = 16$

$a_n = a_1 + (n - 1)d$

$16 = -4 + 4d$

$d = 5$

$a_n = a_1 + (n - 1)d = -4 + (n - 1)5$

$= 5n - 9$

29. $a_3 = 94,\ a_6 = 85$

$a_6 = a_3 + 3d \implies 85 = 94 + 3d \implies d = -3$

$a_1 = a_3 - 2d \implies a_1 = 94 - 2(-3) = 100$

$a_n = a_1 + (n - 1)d = 100 + (n - 1)(-3)$

$= -3n + 103$

30. $a_5 = 190, a_{10} = 115$

$a_{10} = a_5 + 5d \implies 115 = 190 + 5d \implies d = -15$

$a_1 = a_5 - 4d \implies a_1 = 190 - 4(-15) = 250$

$a_n = a_1 + (n - 1)d = 250 + (n - 1)(-15)$

$= -15n + 265$

31. $a_1 = 5,\ d = 6$

$a_1 = 5$

$a_2 = 5 + 6 = 11$

$a_3 = 11 + 6 = 17$

$a_4 = 17 + 6 = 23$

$a_5 = 23 + 6 = 29$

32. $a_1 = 5, d = -\frac{3}{4}$

$a_1 = 5$

$a_2 = 5 - \frac{3}{4} = \frac{17}{4}$

$a_3 = \frac{17}{4} - \frac{3}{4} = \frac{14}{4} = \frac{7}{2}$

$a_4 = \frac{7}{2} - \frac{3}{4} = \frac{11}{4}$

$a_5 = \frac{11}{4} - \frac{3}{4} = \frac{8}{4} = 2$

33. $a_1 = -2.6,\ d = -0.4$

$a_1 = -2.6$

$a_2 = -2.6 + (-0.4) = -3.0$

$a_3 = -3.0 + (-0.4) = -3.4$

$a_4 = -3.4 + (-0.4) = -3.8$

$a_5 = -3.8 + (-0.4) = -4.2$

34. $a_1 = 16.5, d = 0.25$

$a_1 = 16.5$

$a_2 = 16.5 + 0.25 = 16.75$

$a_3 = 16.75 + 0.25 = 17$

$a_4 = 17 + 0.25 = 17.25$

$a_5 = 17.25 + 0.25 = 17.5$

35. $a_1 = 2,\ a_{12} = 46$

$46 = 2 + (12 - 1)d$

$44 = 11d$

$4 = d$

$a_1 = 2$

$a_2 = 2 + 4 = 6$

$a_3 = 6 + 4 = 10$

$a_4 = 10 + 4 = 14$

$a_5 = 14 + 4 = 18$

36. $a_4 = 16, a_{10} = 46$

$16 = a_4 = a_1 + (n - 1)d = a_1 + 3d$

$46 = a_{10} = a_1 + (n - 1)d = a_1 + 9d$

Answer: $a_1 = 1, d = 5$

$a_1 = 1$

$a_2 = 1 + 5 = 6$

$a_3 = 6 + 5 = 11$

$a_4 = 11 + 5 = 16$

$a_5 = 16 + 5 = 21$

37. $a_8 = 26,\ a_{12} = 42$

$a_{12} = a_8 + 4d$

$42 = 26 + 4d \Longrightarrow d = 4$

$a_8 = a_1 + 7d$

$26 = a_1 + 28 \Longrightarrow a_1 = -2$

$a_1 = -2$

$a_2 = -2 + 4 = 2$

$a_3 = 2 + 4 = 6$

$a_4 = 6 + 4 = 10$

$a_5 = 10 + 4 = 14$

38. $a_3 = 19, a_{15} = -1.7$

$19 = a_3 = a_1 + (n - 1)d = a_1 + 2d$

$-1.7 = a_{15} = a_1 + (n - 1)d = a_1 + 14d$

Answer: $a_1 = 22.45, d = -1.725$

$a_1 = 22.45$

$a_2 = 22.45 - 1.725 = 20.725$

$a_3 = 20.725 - 1.725 = 19$

$a_4 = 19 - 1.725 = 17.275$

$a_5 = 17.275 - 1.725 = 15.55$

39. $a_1 = 15,\ a_{k+1} = a_k + 4$

$a_2 = 15 + 4 = 19$

$a_3 = 19 + 4 = 23$

$a_4 = 23 + 4 = 27$

$a_5 = 27 + 4 = 31$

$d = 4$

$c = a_1 - d = 15 - 4 = 11$

$a_n = 4n + 11$

40. $a_1 = 6, a_{k+1} = a_k + 5$

$a_2 = 6 + 5 = 11$

$a_3 = 11 + 5 = 16$

$a_4 = 16 + 5 = 21$

$a_5 = 21 + 5 = 26$

$d = 5$

$a_n = dn + c$

$a_n = 5n + c$

$c = a_1 - d$

$= 6 - 5$

$= 1$

So, $a_n = 5n + 1$.

41. $a_1 = 200,\ a_{k+1} = a_k - 10$

$a_2 = 200 - 10 = 190$

$a_3 = 190 - 10 = 180$

$a_4 = 180 - 10 = 170$

$a_5 = 170 - 10 = 160$

$d = -10$

$c = a_1 - d = 200 - (-10) = 210$

$a_n = -10n + 210$

42. $a_1 = 72, a_{k+1} = a_k - 6$

$a_2 = 72 - 6 = 66$

$a_3 = 66 - 6 = 60$

$a_4 = 60 - 6 = 54$

$a_5 = 54 - 6 = 48$

$d = -6$

$a_n = dn + c$

$a_n = -6n + c$

$c = a_1 - d$

$= 72 - (-6)$

$= 78$

So, $a_n = -6n + 78$.

43. $a_1 = \frac{5}{8}, a_{k+1} = a_k - \frac{1}{8}$

$a_1 = \frac{5}{8}$

$a_2 = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$

$a_3 = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

$a_4 = \frac{3}{8} - \frac{1}{8} = \frac{1}{4}$

$a_5 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$

$d = -\frac{1}{8}$

$c = a_1 - d = \frac{5}{8} - \left(-\frac{1}{8}\right) = \frac{3}{4}$

$a_n = -\frac{1}{8}n + \frac{3}{4}$

44. $a_1 = 0.375, a_{k+1} = a_k + 0.25$

$a_2 = 0.375 + 0.25 = 0.625$

$a_3 = 0.625 + 0.25 = 0.875$

$a_4 = 0.875 + 0.25 = 1.125$

$a_5 = 1.125 + 0.25 = 1.375$

$d = 0.25$

$a_n = dn + c$

$a_n = 0.25n + c$

$c = a_1 - d$

$= 0.375 - 0.25$

$= 0.125$

So, $a_n = 0.25n + 0.125$.

45. $a_1 = 5, a_2 = 11 \implies d = 11 - 5 = 6$

$a_n = a_1 + (n-1)d \implies a_{10} = 5 + 9(6) = 59$

46. $a_1 = 3, a_2 = 13$

$d = a_2 - a_1 = 13 - 3 = 10$

$a_n = dn + c, a_n = 10n + c$

$c = a_1 - d = 3 - 10 = -7$

$a_n = 10n - 7, a_9 = 10(9) - 7 = 83$

47. $a_1 = 4.2, a_2 = 6.6 \implies d = 6.6 - 4.2 = 2.4$

$a_n = a_1 + (n-1)d \implies a_7 = 4.2 + 6(2.4) = 18.6$

48. $a_1 = -0.7, a_2 = -13.8$

$d = a_2 - a_1 = -13.8 + 0.7 = -13.1$

$a_n = dn + c, a_n = -13.1n + c$

$c = a_1 - d = -0.7 + 13.1 = 12.4$

$a_n = -13.1n + 12.4, a_8 = -92.4$

49. $a_n = -\frac{3}{4}n + 8$

$d = -\frac{3}{4}$ so the sequence is decreasing and $a_1 = 7\frac{1}{4}$.

Matches (b).

50. $a_n = 3n - 5$

$d = 3$ so the sequence is increasing and $a_1 = -2$.

Matches (d).

51. $a_n = 2 + \frac{3}{4}n$

$d = \frac{3}{4}$ so the sequence is increasing and $a_1 = 2\frac{3}{4}$.

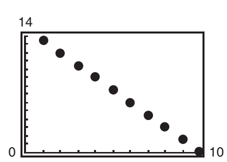
Matches (c).

52. $a_n = 25 - 3n$

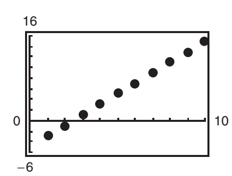
$d = -3$ so the sequence is decreasing and $a_1 = 22$.

Matches (a).

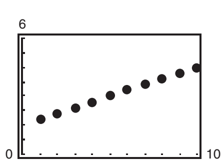
53. $a_n = 15 - \frac{3}{2}n$



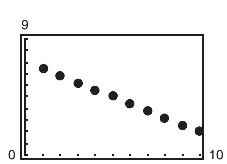
54. $a_n = -5 + 2n$



55. $a_n = 0.2n + 3$



56. $a_n = -0.3n + 8$



57. 8, 20, 32, 44, . . .

$a_1 = 8, \ d = 12, \ n = 10$

$a_{10} = 8 + 9(12) = 116$

$S_{10} = \frac{10}{2}(8 + 116) = 620$

58. $2, 8, 14, 20, \ldots, n = 25$

$d = 6, c = 2 - 6 = -4$

$a_n = 6n - 4$

$a_1 = 2$ and $a_{25} = 146$

$S_{25} = \frac{25}{2}(2 + 146) = 1850$

59. $4.2, 3.7, 3.2, 2.7, \ldots$

$a_1 = 4.2, d = -0.5, n = 12$

$a_{12} = 4.2 + 11(-0.5) = -1.3$

$S_{12} = \frac{12}{2}[4.2 + (-1.3)] = 17.4$

60. $0.5, 0.9, 1.3, 1.7, \ldots, n = 10$

$d = 0.4, c = 0.1$

$a_n = 0.4n + 0.1$

$a_1 = 0.5$ and $a_{10} = 4.1$

$S_{10} = \frac{10}{2}(0.5 + 4.1) = 23$

61. $40, 37, 34, 31, \ldots$

$a_1 = 40, d = -3, n = 10$

$a_{10} = 40 + 9(-3) = 13$

$S_{10} = \dfrac{10}{2}(40 + 13) = 265$

62. $75, 70, 65, 60, \ldots, n = 25$

$d = -5, c = 80$

$a_n = -5n + 80$

$a_1 = 75$ and $a_{25} = -45$

$S_{25} = \dfrac{25}{2}(75 - 45) = 375$

63. $a_1 = 100, \ a_{25} = 220, \ n = 25$

$S_n = \dfrac{n}{2}[a_1 + a_n]$

$S_{25} = \dfrac{25}{2}(100 + 220) = 4000$

64. $a_1 = 15, a_{100} = 307, n = 100$

$S_{100} = \frac{100}{2}(15 + 307) = 16{,}100$

65. $a_n = 2n - 1$

$a_1 = 1, a_{100} = 199$

$\sum_{n=1}^{100}(2n - 1) = \frac{100}{2}(1 + 199) = 10{,}000$

66. $a_0 = -10, a_{60} = 50, n = 60$

$\sum_{i=0}^{60}(i - 10) = \frac{60}{2}(-10 + 50)$

$= 1200$

67. $a_1 = 1, \ a_{50} = 50, \ n = 50$

$\sum_{n=1}^{50} n = \frac{50}{2}(1 + 50) = 1275$

68. $a_n = 2n$

$a_1 = 2, a_{100} = 200, n = 100$

$\sum_{n=1}^{100} 2n = \frac{100}{2}(2 + 200) = 10{,}100$

69. $a_{10} = 60, a_{100} = 600, n = 91$

$\sum_{n=10}^{100} 6n = \frac{91}{2}(60 + 600) = 30{,}030$

70. $a_n = 7n$

$a_{51} = 357, a_{100} = 700$

$\sum_{n=51}^{100} 7n = \frac{50}{2}(357 + 700) = 26{,}425$

71. $\sum_{n=11}^{30} n - \sum_{n=1}^{10} n = \frac{20}{2}(11 + 30) - \frac{10}{2}(1 + 10) = 355$

72. $\sum_{n=51}^{100} n - \sum_{n=1}^{50} n = \frac{50}{2}(51 + 100) - \frac{50}{2}(1 + 50)$

$= 3775 - 1275 = 2500$

73. $a_1 = 1, a_{400} = 799, n = 400$

$\sum_{n=1}^{400}(2n - 1) = \frac{400}{2}(1 + 799) = 160{,}000$

74. $a_n = 1000 - n$

$a_1 = 999, a_{250} = 750, n = 250$

$\sum_{n=1}^{250}(1000 - n) = \frac{250}{2}(999 + 750) = 218{,}625$

75. $\sum_{n=1}^{20}(2n + 5) = 520$

76. $a_0 = 1000, \ a_{50} = 750, \ n = 51$

$\sum_{n=0}^{50}(1000 - 5n) = \frac{51}{2}(1000 + 750) = 44{,}625$

77. $\sum_{n=1}^{100} \dfrac{n + 4}{2} = 2725$

78. $a_0 = \dfrac{1}{2}, a_{100} = \dfrac{-73}{4}, n = 101$

$\sum_{n=0}^{100} \dfrac{8 - 3n}{16} = \dfrac{101}{2}\left(\dfrac{1}{2} - \dfrac{73}{4}\right) = -896.375$

79. $\sum_{i=1}^{60}\left(250 - \frac{8}{3}i\right) = 10{,}120$

80. $a_1 = 4.525, a_{200} = 9.5, n = 200$

$\sum_{j=1}^{200} (4.5 + 0.025j) = \frac{200}{2}(4.525 + 9.5) = 1402.5$

81. (a) $a_1 = 32{,}500,\ d = 1500$

$a_6 = a_1 + 5d = 32{,}500 + 5(1500) = \$40{,}000$

(b) $S_6 = \frac{6}{2}[32{,}500 + 40{,}000] = \$217{,}500$

82. (a) $a_1 = 36{,}800, d = 1750$

$a_6 = a_1 + 5d = 36{,}800 + 5(1750) = \$45{,}550$

(b) $S_6 = \frac{6}{2}[36{,}800 + 45{,}550] = \$247{,}050$

83. $a_1 = 20,\ d = 4,\ n = 30$

$a_{30} = 20 + 29(4) = 136$

$S_{30} = \frac{30}{2}(20 + 136) = 2340$ seats

84. $a_1 = 15, d = 3, n = 36$

$a_{36} = 15 + 35(3) = 120$

$S_{36} = \frac{36}{2}(15 + 120) = 2430$ seats

85. $a_1 = 14, a_{18} = 31$

$S_{18} = \frac{18}{2}(14 + 31) = 405$ bricks

86. $a_1 = 14, a_{28} = 0.5, n = 28$

$S_{28} = \frac{28}{2}(14 + 0.5) = 203$ bricks

87. 4.9, 14.7, 24.5, 34.3, . . .

$d = 9.8$

$a_{10} = 4.9 + 9(9.8) = 93.1$ meters

$S_{10} = \frac{10}{2}(4.9 + 93.1) = 490$ meters

88. $a_1 = 16, a_2 = 48, a_3 = 80, a_4 = 112$

$d = 32$

$a_n = dn + c = 32n + c$

$c = a_1 - d = 16 - 32 = -16$

$a_n = 32n - 16$

Distance $= \sum_{n=1}^{7}(32n - 16) = 784$ ft

89. (a) $a_1 = 200, a_2 = 175 \implies d = -25$

$c = 200 - (-25) = 225$

$a_n = -25n + 225$

(b) $a_8 = -25(8) + 225 = 25$

$S_8 = \frac{8}{2}(200 + 25) = \$900$

90. (a) $a_1 = 1200, a_2 = 1100, a_3 = 1000$

$d = -100$

$a_n = dn + c$

$a_n = -100n + c$

$c = a_1 - d = 1200 + 100 = 1300$

$a_n = -100n + 1300$

(b) Total prize money $= \sum_{n=1}^{12}(-100n + 1300)$

$= \frac{12}{2}(1200 + 100)$

$= \$7800$

91. $a_n = 1500n + 6500$

$a_1 = 8000, a_6 = 15{,}500$

$S_6 = \frac{6}{2}(8000 + 15{,}500) = \$70{,}500$

The cost of gasoline, labor, equipment, insurance, and maintenance are a few economic factors that could prevent the company from meeting its goals, but the biggest unknown variable is the amount of annual snowfall.

92. $a_1 = 15{,}000$

$d = 5{,}000$

$n = 1, \ldots, 10$

$a_n = dn + c = 5000n + c$

$c = a_1 - d = 15{,}000 - 5000 = 10{,}000$

$a_n = 5000n + 10{,}000$

$\text{Total sales} = \sum_{n=1}^{10}(5000n + 10{,}000) = \frac{10}{2}(15{,}000 + 60{,}000) = \$375{,}000$

93. (a)

Monthly Payment	Unpaid Balance
$a_1 = 200 + 0.01(2000) = \$220$	\$1800
$a_2 = 200 + 0.01(1800) = \$218$	\$1600
$a_3 = 200 + 0.01(1600) = \$216$	\$1400
$a_4 = 200 + 0.01(1400) = \$214$	\$1200
$a_5 = 200 + 0.01(1200) = \$212$	\$1000
$a_6 = 200 + 0.01(1000) = \$210$	\$800

(b) $a_n = -2n + 222 \implies a_{10} = 202$

$S_{10} = \frac{10}{2}(220 + 202) = \$2110$

Interest paid: \$110

94. (a) Borrowed Amount $= a_0 = \$5{,}000$

Amount of Balance Paid Per Month $= \$250$

Unpaid Balance $= a_n = 5000 - 250n$

Interest $= I =$ Balance Before Payment $\cdot$ 1% $= a_{n-1} \cdot 0.01$

Total Payment $= \$250 + I$

Month (n)	1	2	3	4	5	6
Interest (I)	\$50	\$47.50	\$45.00	\$42.50	\$40.00	\$37.50
Total Payment (\$250 + I)	\$300	\$297.50	\$295.00	\$292.50	\$290.00	\$287.50
Unpaid Balance (a_n)	\$4750	\$4500	\$4250	\$4000	\$3750	\$3500

Month (n)	7	8	9	10	11	12
Interest (I)	\$35.00	\$32.50	\$30.00	\$27.50	\$25.00	\$22.50
Total Payment (\$250 + I)	\$285.00	\$282.50	\$280.00	\$277.50	\$275.00	\$272.50
Unpaid Balance (a_n)	\$3250	\$3000	\$2750	\$2500	\$2250	\$2000

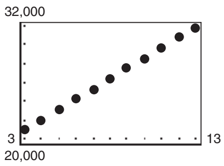
(b) $\text{Total Interest Paid} = \sum_{n=1}^{20}[5000 - 250(n-1)] \cdot 0.01 = \frac{20}{2}[(5000)(0.01) + (250)(0.01)] = \$525$

95. (a) Using (5, 23,078) and (6, 24,176) we have $d = 1098$ and $c = 23{,}078 - 5(1098) = 17{,}588$.

$a_n \approx 1098n + 17{,}588$

(b) $a_n \approx 1114.95n + 17{,}795.07$

The models are similar.

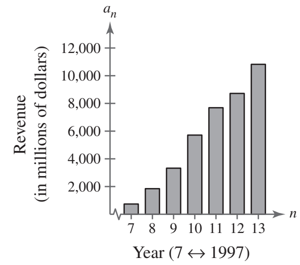
(c) 

(d) For 2004 use $n = 14$: \$32,960

For 2005 use $n = 15$: \$34,058

(e) Answers will vary.

96. (a) $n = 7$ is 1997.



(b) $a_n = \text{Revenue} = 1726.93n - 11{,}718.43$

(c) $\text{Total revenue} = \sum_{n=7}^{13} (1726.93n - 11{,}718.43)$

$= \frac{7}{2}(370.08 + 10{,}731.66)$

$= \$38{,}856$ million

(d) $a_{18} = 1726.93(18) - 11{,}718.43 = \$19{,}366.31$ million

97. True; given a_1 and a_2 then $d = a_2 - a_1$ and $a_n = a_1 + (n - 1)d$.

98. True, by the formula for the sum of a finite arithmetic sequence,

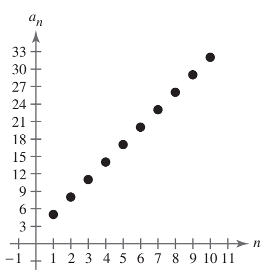
$S_n = \frac{n}{2}(a_1 + a_n)$.

99. A sequence is arithmetic if the differences between consecutive terms are the same.

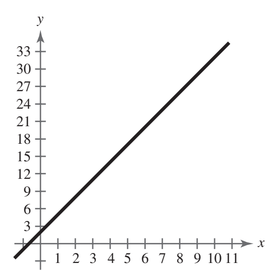
$a_{n+1} - a_n = d$ for $n \geq 1$

100. First term plus $(n - 1)$ times the common difference

101. (a) $a_n = 2 + 3n$



(b) $y = 3x + 2$



(c) The graph of $a_n = 2 + 3n$ contains only points at the positive integers. The graph of $y = 3x + 2$ is a solid line which contains these points.

(d) The slope $m = 3$ is equal to the common difference $d = 3$. In general, these should be equal.

102. (a) $1 + 3 = 4$

$1 + 3 + 5 = 9$

$1 + 3 + 5 + 7 = 16$

$1 + 3 + 5 + 7 + 9 = 25$

$1 + 3 + 5 + 7 + 9 + 11 = 36$

(b) $S_n = n^2$

$S_7 = 1 + 3 + 5 + 7 + 9 + 11 + 13 = 49 = 7^2$

(c) $S_n = \frac{n}{2}[1 + (2n - 1)] = \frac{n}{2}(2n) = n^2$

103. $S_{20} = \frac{20}{2}\{a_1 + [a_1 + (20 - 1)(3)]\} = 650$

$10(2a_1 + 57) = 650$

$2a_1 + 57 = 65$

$2a_1 = 8$

$a_1 = 4$

104. Let $S_n = \frac{n}{2}(a_1 + a_n)$ be the sum of the first n terms of the original sequence.

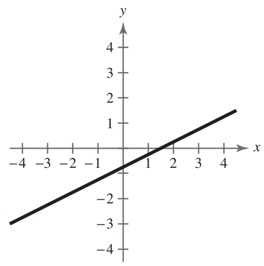
$$\begin{aligned} S_n{}' &= \frac{n}{2}(a_1 + 5 + a_n + 5) \\ &= \frac{n}{2}(a_1 + a_n + 10) \\ &= \frac{n}{2}(a_1 + a_n) + \frac{n}{2}(10) \\ &= \frac{n}{2}(a_1 + a_n) + 5n \\ &= S_n + 5n \end{aligned}$$

105. $2x - 4y = 3$

$y = \frac{1}{2}x - \frac{3}{4}$

Slope: $m = \frac{1}{2}$

y-intercept: $\left(0, -\frac{3}{4}\right)$

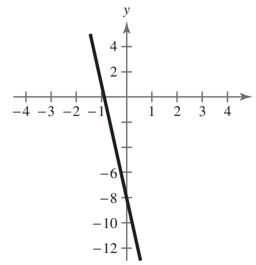


106. $9x + y = -8$

$y = -9x - 8$

Slope: -9

y-intercept: $(0, -8)$



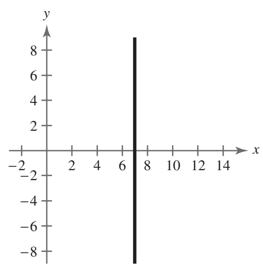
107. $x - 7 = 0$

$x = 7$

Vertical line

No slope

No y-intercept

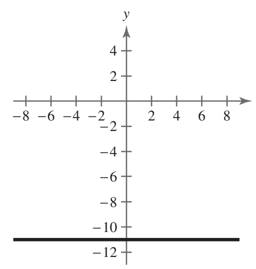


108. $y + 11 = 0$

$y = -11$

Slope: 0

y-intercept: $(0, -11)$



109. $\begin{cases} 2x - y + 7z = -10 & \text{Equation 1} \\ 3x + 2y - 4z = 17 & \text{Equation 2} \\ 6x - 5y + z = -20 & \text{Equation 3} \end{cases}$

$\begin{cases} x - \frac{1}{2}y + \frac{7}{2}z = -5 & \frac{1}{2}\text{Eq.1} \\ 3x + 2y - 4z = 17 \\ 6x - 5y + z = -20 \end{cases}$

$\begin{cases} x - \frac{1}{2}y + \frac{7}{2}z = -5 \\ \frac{7}{2}y - \frac{29}{2}z = 32 & (-3)\text{Eq.1} + \text{Eq.2} \\ -2y - 20z = 10 & (-6)\text{Eq.1} + \text{Eq.3} \end{cases}$

$\begin{cases} x - \frac{1}{2}y + \frac{7}{2}z = -5 \\ -2y - 20z = 10 \\ \frac{7}{2}y - \frac{29}{2}z = 32 \end{cases}$

$\begin{cases} x - \frac{1}{2}y + \frac{7}{2}z = -5 \\ y + 10z = -5 & \left(-\frac{1}{2}\right)\text{Eq.2} \\ 7y - 29z = 64 & 2\text{ Eq.3} \end{cases}$

$\begin{cases} x + \frac{17}{2}z = -\frac{15}{2} & \left(\frac{1}{2}\right)\text{Eq.2} + \text{Eq.1} \\ y + 10z = -5 \\ -99z = 99 & (-7)\text{Eq.2} + \text{Eq.3} \end{cases}$

$\begin{cases} x + \frac{17}{2}z = -\frac{15}{2} \\ y + 10z = -5 \\ z = -1 & \left(-\frac{1}{99}\right)\text{Eq.3} \end{cases}$

$\begin{cases} x = 1 & \left(-\frac{17}{2}\right)\text{Eq.3} + \text{Eq.1} \\ y = 5 & (-10)\text{Eq.3} + \text{Eq.2} \\ z = -1 \end{cases}$

Answer: $x = 1, y = 5, z = -1$

110. $\left[\begin{array}{rrrcr} -1 & 4 & 10 & \vdots & 4 \\ 5 & -3 & 1 & \vdots & 31 \\ 8 & 2 & -3 & \vdots & -5 \end{array}\right]$

$5R_1 + R_2 \rightarrow \left[\begin{array}{rrrcr} -1 & 4 & 10 & \vdots & 4 \\ 0 & 17 & 51 & \vdots & 51 \\ 8 & 2 & -3 & \vdots & -5 \end{array}\right]$

$8R_1 + R_3 \rightarrow \left[\begin{array}{rrrcr} -1 & 4 & 10 & \vdots & 4 \\ 0 & 17 & 51 & \vdots & 51 \\ 0 & 34 & 77 & \vdots & 27 \end{array}\right]$

$2R_2 - R_3 \rightarrow \left[\begin{array}{rrrcr} -1 & 4 & 10 & \vdots & 4 \\ 0 & 17 & 51 & \vdots & 51 \\ 0 & 0 & 25 & \vdots & 75 \end{array}\right]$

$-R_1 \rightarrow \left[\begin{array}{rrrcr} 1 & -4 & -10 & \vdots & -4 \\ 0 & 17 & 51 & \vdots & 51 \\ 0 & 0 & 25 & \vdots & 75 \end{array}\right]$

$\frac{1}{17}R_2 \rightarrow \left[\begin{array}{rrrr} 1 & -4 & -10 & -4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 25 & 75 \end{array}\right]$

$\frac{1}{25}R_3 \rightarrow \left[\begin{array}{rrrcr} 1 & -4 & -10 & \vdots & -4 \\ 0 & 1 & 3 & \vdots & 3 \\ 0 & 0 & 1 & \vdots & 3 \end{array}\right]$

$R_2 - 3R_3 \rightarrow \left[\begin{array}{rrrcr} 1 & -4 & -10 & \vdots & -4 \\ 0 & 1 & 0 & \vdots & -6 \\ 0 & 0 & 1 & \vdots & 3 \end{array}\right]$

$R_1 + 4R_2 + 10R_3 \rightarrow \left[\begin{array}{rrrcr} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & -6 \\ 0 & 0 & 1 & \vdots & 3 \end{array}\right]$

$x = 2, y = -6, z = 3$

111. Answers will vary.

Section 9.3 Geometric Sequences and Series

- You should be able to identify a geometric sequence, find its common ratio, and find the nth term.
- You should know that the nth term of a geometric sequence with common ratio r is given by $a_n = a_1 r^{n-1}$.
- You should know that the nth partial sum of a geometric sequence with common ratio $r \neq 1$ is given by

  $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$.
- You should know that if $|r| < 1$, then

  $$\sum_{n=1}^{\infty} a_1 r^{n-1} = \sum_{n=0}^{\infty} a_1 r^n = \frac{a_1}{1 - r}.$$

Vocabulary Check

1. geometric; common

2. $a_n = a_1 r^{n-1}$

3. $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$

4. geometric series

5. $S = \frac{a_1}{1 - r}$

1. 5, 15, 45, 135, . . .

Geometric sequence, $r = 3$

2. 3, 12, 48, 192, . . .

Geometric sequence, $r = 4$

3. 3, 12, 21, 30, . . .

Not a geometric sequence

Note: It is an arithmetic sequence with $d = 9$.

4. 36, 27, 18, 9, . . .

Not a geometric sequence

5. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

Geometric sequence, $r = -\frac{1}{2}$

6. 5, 1, 0.2, 0.04, . . .

Geometric sequence, $r = \frac{1}{5} = 0.2$

7. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \ldots$

Geometric sequence, $r = 2$

8. $9, -6, 4, -\frac{8}{3}, \ldots$

Geometric sequence, $r = -\frac{2}{3}$

9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Not a geometric sequence

10. $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \frac{4}{11}, \ldots$

Not a geometric sequence

11. $a_1 = 2,\ r = 3$

$a_1 = 2$

$a_2 = 2(3) = 6$

$a_3 = 6(3) = 18$

$a_4 = 18(3) = 54$

$a_5 = 54(3) = 162$

12. $a_1 = 6, r = 2$

$a_1 = 6$

$a_2 = 6(2)^1 = 12$

$a_3 = 6(2)^2 = 24$

$a_4 = 6(2)^3 = 48$

$a_5 = 6(2)^4 = 96$

13. $a_1 = 1,\ r = \frac{1}{2}$

$a_1 = 1$

$a_2 = 1\left(\frac{1}{2}\right) = \frac{1}{2}$

$a_3 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$a_4 = \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8}$

$a_5 = \frac{1}{8}\left(\frac{1}{2}\right) = \frac{1}{16}$

14. $a_1 = 1, r = \frac{1}{3}$

$a_1 = 1$

$a_2 = 1\left(\frac{1}{3}\right)^1 = \frac{1}{3}$

$a_3 = 1\left(\frac{1}{3}\right)^2 = \frac{1}{9}$

$a_4 = 1\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

$a_5 = 1\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

15. $a_1 = 5,\ r = -\frac{1}{10}$

$a_1 = 5$

$a_2 = 5\left(-\frac{1}{10}\right) = -\frac{1}{2}$

$a_3 = \left(-\frac{1}{2}\right)\left(-\frac{1}{10}\right) = \frac{1}{20}$

$a_4 = \frac{1}{20}\left(-\frac{1}{10}\right) = -\frac{1}{200}$

$a_5 = \left(-\frac{1}{200}\right)\left(-\frac{1}{10}\right) = \frac{1}{2000}$

16. $a_1 = 6, r = -\frac{1}{4}$

$a_1 = 6$

$a_2 = 6\left(-\frac{1}{4}\right)^1 = -\frac{3}{2}$

$a_3 = 6\left(-\frac{1}{4}\right)^2 = \frac{3}{8}$

$a_4 = 6\left(-\frac{1}{4}\right)^3 = -\frac{3}{32}$

$a_5 = 6\left(-\frac{1}{4}\right)^4 = \frac{3}{128}$

17. $a_1 = 1,\ r = e$

$a_1 = 1$

$a_2 = 1(e) = e$

$a_3 = (e)(e) = e^2$

$a_4 = (e^2)(e) = e^3$

$a_5 = (e^3)(e) = e^4$

18. $a_1 = 3, r = \sqrt{5}$

$a_1 = 3$

$a_2 = 3\left(\sqrt{5}\right)^1 = 3\sqrt{5}$

$a_3 = 3\left(\sqrt{5}\right)^2 = 15$

$a_4 = 3\left(\sqrt{5}\right)^3 = 15\sqrt{5}$

$a_5 = 3\left(\sqrt{5}\right)^4 = 75$

19. $a_1 = 2, r = \dfrac{x}{4}$

$a_1 = 2$

$a_2 = 2\left(\dfrac{x}{4}\right) = \dfrac{x}{2}$

$a_3 = \left(\dfrac{x}{2}\right)\left(\dfrac{x}{4}\right) = \dfrac{x^2}{8}$

$a_4 = \left(\dfrac{x^2}{8}\right)\left(\dfrac{x}{4}\right) = \dfrac{x^3}{32}$

$a_5 = \left(\dfrac{x^3}{32}\right)\left(\dfrac{x}{4}\right) = \dfrac{x^4}{128}$

20. $a_1 = 5, r = 2x$

$a_1 = 5$

$a_2 = 5(2x)^1 = 10x$

$a_3 = 5(2x)^2 = 20x^2$

$a_4 = 5(2x)^3 = 40x^3$

$a_5 = 5(2x)^4 = 80x^4$

21. $a_1 = 64,\ a_{k+1} = \dfrac{1}{2}a_k$

$a_1 = 64$

$a_2 = \dfrac{1}{2}(64) = 32$

$a_3 = \dfrac{1}{2}(32) = 16$

$a_4 = \dfrac{1}{2}(16) = 8$

$a_5 = \dfrac{1}{2}(8) = 4$

$r = \dfrac{1}{2}$

$a_n = 64\left(\dfrac{1}{2}\right)^{n-1} = 128\left(\dfrac{1}{2}\right)^n$

22. $a_1 = 81, a_{k+1} = \frac{1}{3}a_k$

$a_1 = 81$

$a_2 = \frac{1}{3}(81) = 27$

$a_3 = \frac{1}{3}(27) = 9$

$a_4 = \frac{1}{3}(9) = 3$

$a_5 = \frac{1}{3}(3) = 1$

$a_n = 81\left(\frac{1}{3}\right)^{n-1} = 243\left(\frac{1}{3}\right)^n$

23. $a_1 = 7, a_{k+1} = 2a_k$

$a_1 = 7$

$a_2 = 2(7) = 14$

$a_3 = 2(14) = 28$

$a_4 = 2(28) = 56$

$a_5 = 2(56) = 112$

$r = 2$

$a_n = 7(2)^{n-1} = \frac{7}{2}(2)^n$

24. $a_1 = 5, a_{k+1} = -2a_k$

$a_1 = 5$

$a_2 = -2(5) = -10$

$a_3 = -2(-10) = 20$

$a_4 = -2(20) = -40$

$a_5 = -2(-40) = 80$

$a_n = 5(-2)^{n-1} = -\frac{5}{2}(-2)^n$

25. $a_1 = 6,\ a_{k+1} = -\frac{3}{2}a_k$

$a_1 = 6$

$a_2 = -\frac{3}{2}(6) = -9$

$a_3 = -\frac{3}{2}(-9) = \frac{27}{2}$

$a_4 = -\frac{3}{2}\left(\frac{27}{2}\right) = -\frac{81}{4}$

$a_5 = -\frac{3}{2}\left(-\frac{81}{4}\right) = \frac{243}{8}$

$r = -\frac{3}{2}$

$a_n = 6\left(-\frac{3}{2}\right)^{n-1}$ or $a_n = -4\left(-\frac{3}{2}\right)^n$

26. $a_1 = 48, a_{k+1} = -\frac{1}{2}a_k$

$a_1 = 48$

$a_2 = -\frac{1}{2}(48) = -24$

$a_3 = -\frac{1}{2}(-24) = 12$

$a_4 = -\frac{1}{2}(12) = -6$

$a_5 = -\frac{1}{2}(-6) = 3$

$a_n = 48\left(-\frac{1}{2}\right)^{n-1} = -96\left(-\frac{1}{2}\right)^n$

27. $a_1 = 4,\ r = \frac{1}{2},\ n = 10$

$a_n = a_1r^{n-1} = 4\left(\frac{1}{2}\right)^{n-1}$

$a_{10} = 4\left(\frac{1}{2}\right)^9 = \left(\frac{1}{2}\right)^7 = \frac{1}{128}$

28. $a_1 = 5,\ r = \dfrac{3}{2},\ n = 8$

$a_n = a_1r^{n-1} = 5\left(\dfrac{3}{2}\right)^{n-1}$

$a_8 = 5\left(\dfrac{3}{2}\right)^7 = \dfrac{10{,}935}{128}$

29. $a_1 = 6,\ r = -\dfrac{1}{3},\ n = 12$

$a_n = a_1r^{n-1} = 6\left(-\dfrac{1}{3}\right)^{n-1}$

$a_{12} = 6\left(-\dfrac{1}{3}\right)^{11} = -\dfrac{2}{3^{10}}$

30. $a_1 = 64, r = -\dfrac{1}{4}, n = 10$

$a_n = a_1r^{n-1} = 64\left(-\dfrac{1}{4}\right)^{n-1}$

$a_{10} = 64\left(-\dfrac{1}{4}\right)^9 = -\dfrac{64}{262{,}144}$

31. $a_1 = 100,\ r = e^x,\ n = 9$

$a_n = a_1r^{n-1} = 100(e^x)^{n-1}$

$a_9 = 100(e^x)^8 = 100e^{8x}$

32. $a_1 = 1, r = \sqrt{3}, n = 8$

$a_n = a_1r^{n-1} = 1\left(\sqrt{3}\right)^{n-1}$

$a_8 = 1\left(\sqrt{3}\right)^7 = 27\sqrt{3}$

33. $a_1 = 500,\ r = 1.02,\ n = 40$

$a_n = a_1r^{n-1} = 500(1.02)^{n-1}$

$a_{40} = 500(1.02)^{39} \approx 1082.372$

34. $a_1 = 1000,\ r = 1.005,\ n = 60$

$a_n = a_1r^{n-1} = 1000(1.005)^{n-1}$

$a_{60} = 1000(1.005)^{59} \approx 1342.139$

35. $7, 21, 63, \ldots \Rightarrow r = 3$

$a_n = 7(3)^{n-1}$

$a_9 = 7(3)^8 = 45{,}927$

36. $a_1 = 3, a_2 = 36, a_3 = 432$

$r = \dfrac{a_2}{a_1} = \dfrac{36}{3} = 12$

$a_n = a_1r^{(n-1)}$

$a_7 = (3)(12)^6 = 8{,}957{,}952$

37. $5, 30, 180, \ldots \Rightarrow r = 6$

$a_n = 5(6)^{n-1}$

$a_{10} = 5(6)^9 = 50{,}388{,}480$

38. $a_1 = 4, a_2 = 8, a_3 = 16$

$r = \dfrac{a_2}{a_1} = \dfrac{8}{4} = 2$

$a_n = a_1r^{n-1}$

$a_{22} = (4)(2)^{21} = 8{,}388{,}608$

39. $a_1 = 16, a_4 = \dfrac{27}{4}$

$a_4 = a_1r^3$

$\dfrac{27}{4} = 16r^3$

$\dfrac{27}{64} = r^3$

$\dfrac{3}{4} = r$

$a_n = 16\left(\dfrac{3}{4}\right)^{n-1}$

$a_3 = 16\left(\dfrac{3}{4}\right)^2 = 9$

40. $a_2 = 3, a_5 = \dfrac{3}{64}$

$a_5 = a_2 r^{(5-2)}$

$a_5 = a_2 r^3$

$\dfrac{3}{64} = 3r^3$

$\dfrac{1}{64} = r^3$

$r = \dfrac{1}{4}$

$a_2 = a_1 r^1$

$3 = a_1\left(\dfrac{1}{4}\right)$

$a_1 = 12$

41. $a_4 = -18, a_7 = \dfrac{2}{3}$

$a_7 = a_4 r^3$

$\dfrac{2}{3} = -18r^3$

$-\dfrac{1}{27} = r^3$

$-\dfrac{1}{3} = r$

$a_6 = \dfrac{a_7}{r} = \dfrac{2/3}{-1/3} = -2$

42. $a_3 = \dfrac{16}{3}, a_5 = \dfrac{64}{27}$

$a_5 = a_3 r^{(5-3)}$

$a_5 = a_3 r^2$

$\dfrac{64}{27} = \dfrac{16}{3} r^2$

$r^2 = \dfrac{4}{9}$

$r = \dfrac{2}{3}$

$a_7 = a_5 r^{(7-5)}$

$a_7 = a_5 r^2$

$a_7 = \left(\dfrac{64}{27}\right)\left(\dfrac{2}{3}\right)^2 = \dfrac{256}{243}$

43. $a_n = 18\left(\frac{2}{3}\right)^{n-1}$

$a_1 = 18$ and $r = \frac{2}{3}$

Since $0 < r < 1$, the sequence is decreasing.

Matches (a).

44. $a_n = 18\left(-\frac{2}{3}\right)^{n-1}$

$r = \left(-\frac{2}{3}\right) > -1$, so the sequence alternates as it approaches 0.

Matches (c).

45. $a_n = 18\left(\frac{3}{2}\right)^{n-1}$

$a_1 = 18$ and $r = \frac{3}{2} > 1$, so the sequence is increasing.

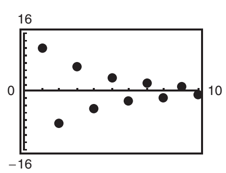
Matches (b).

46. $a_n = 18\left(-\frac{3}{2}\right)^{n-1}$

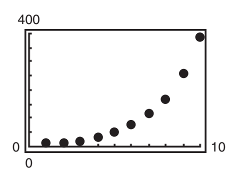
$r = \left(-\frac{3}{2}\right) < -1$, so the sequence alternates as it approaches ∞.

Matches (d).

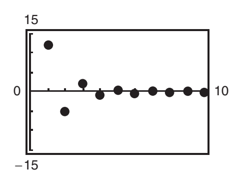
47. $a_n = 12(-0.75)^{n-1}$



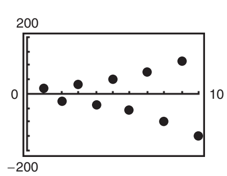
48. $a_n = 10(1.5)^{n-1}$



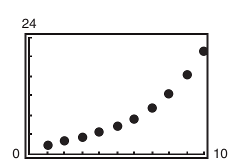
49. $a_n = 12(-0.4)^{n-1}$



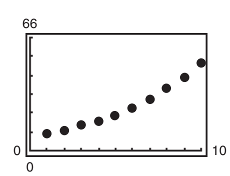
50. $a_n = 20(-1.25)^{n-1}$



51. $a_n = 2(1.3)^{n-1}$



52. $a_n = 10(1.2)^{n-1}$



53. $\displaystyle\sum_{n=1}^{9} 2^{n-1} = 1 + 2^1 + 2^2 + \cdots + 2^8 \implies a_1 = 1,\ r = 2$

$S_9 = \dfrac{1(1 - 2^9)}{1 - 2} = 511$

54. $\sum_{n=1}^{10}\left(\frac{5}{2}\right)^{n-1} = 1 + \left(\frac{5}{2}\right)^1 + \left(\frac{5}{2}\right)^2 + \cdots + \left(\frac{5}{2}\right)^9 \Rightarrow a_1 = 1, r = \frac{5}{2}$

$$S_{10} = 1\left[\frac{1-\left(\frac{5}{2}\right)^{10}}{1-\left(\frac{5}{2}\right)}\right] = -\frac{2}{3}\left[1-\left(\frac{5}{2}\right)^{10}\right] = \frac{3{,}254{,}867}{512} \approx 6357.162$$

55. $\sum_{n=1}^{9}(-2)^{n-1} \Rightarrow a_1 = 1, r = -2, n = 9$

$$S_9 = 1\left(\frac{1-(-2)^9}{1-(-2)}\right) = 171$$

56. $\sum_{n=1}^{8}5\left(-\frac{3}{2}\right)^{n-1} = 5 + 5\left(-\frac{3}{2}\right)^1 + 5\left(-\frac{3}{2}\right)^2 + \cdots + 5\left(-\frac{3}{2}\right)^7 \Rightarrow a_1 = 5, r = -\frac{3}{2}$

$$S_8 = 5\left[\frac{1-\left(-\frac{3}{2}\right)^8}{1-\left(-\frac{3}{2}\right)}\right] = 2\left[1-\left(-\frac{3}{2}\right)^8\right] = -\frac{6305}{128} \approx -49.258$$

57. $\sum_{i=1}^{7}64\left(-\frac{1}{2}\right)^{i-1} = 64 + 64\left(-\frac{1}{2}\right)^1 + 64\left(-\frac{1}{2}\right)^2 + \cdots + 64\left(-\frac{1}{2}\right)^6 \Rightarrow a_1 = 64,\ r = -\frac{1}{2}$

$$S_7 = 64\left[\frac{1-\left(-\frac{1}{2}\right)^7}{1-\left(-\frac{1}{2}\right)}\right] = \frac{128}{3}\left[1-\left(-\frac{1}{2}\right)^7\right] = 43$$

58. $\sum_{i=1}^{10}2\left(\frac{1}{4}\right)^{i-1} = 2 + 2\left(\frac{1}{4}\right)^1 + 2\left(\frac{1}{4}\right)^2 + \cdots + 2\left(\frac{1}{4}\right)^9 \Rightarrow a_1 = 2, r = \frac{1}{4}$

$$S_{10} = 2\left[\frac{1-\left(\frac{1}{4}\right)^{10}}{1-\left(\frac{1}{4}\right)}\right] = \frac{8}{3}\left[1-\left(\frac{1}{4}\right)^{10}\right] \approx 2.667$$

59. $\sum_{i=1}^{6}32\left(\frac{1}{4}\right)^{i-1} = 32 + 32\left(\frac{1}{4}\right)^1 + 32\left(\frac{1}{4}\right)^2 + 32\left(\frac{1}{4}\right)^3 + 32\left(\frac{1}{4}\right)^4 + 32\left(\frac{1}{4}\right)^5 \Rightarrow a_1 = 32, r = \frac{1}{4}, n = 6$

$$S_6 = 32\left(\frac{1-\left(\frac{1}{4}\right)^6}{1-\frac{1}{4}}\right) = \frac{1365}{32}$$

60. $\sum_{i=1}^{12}16\left(\frac{1}{2}\right)^{i-1} = 16 + 16\left(\frac{1}{2}\right)^1 + 16\left(\frac{1}{2}\right)^2 + \cdots + 16\left(\frac{1}{2}\right)^{11} \Rightarrow a_1 = 16, r = \frac{1}{2}$

$$S_{12} = 16\left[\frac{1-\left(\frac{1}{2}\right)^{12}}{1-\left(\frac{1}{2}\right)}\right] = 32\left[1-\left(\frac{1}{2}\right)^{12}\right] = \frac{4095}{128} \approx 31.992$$

61. $\sum_{n=0}^{20}3\left(\frac{3}{2}\right)^n = \sum_{n=1}^{21}3\left(\frac{3}{2}\right)^{n-1} = 3 + 3\left(\frac{3}{2}\right)^1 + 3\left(\frac{3}{2}\right)^2 + \cdots + 3\left(\frac{3}{2}\right)^{20} \Rightarrow a_1 = 3,\ r = \frac{3}{2}$

$$S_{21} = 3\left[\frac{1-\left(\frac{3}{2}\right)^{21}}{1-\frac{3}{2}}\right] = -6\left[1-\left(\frac{3}{2}\right)^{21}\right] \approx 29{,}921.311$$

62. $\sum_{n=0}^{40}5\left(\frac{3}{5}\right)^n = 5 + \sum_{n=1}^{40}5\left(\frac{3}{5}\right)^n = 5 + \left[5\left(\frac{3}{5}\right)^1 + 5\left(\frac{3}{5}\right)^2 + 5\left(\frac{3}{5}\right)^3 + \cdots + 5\left(\frac{3}{5}\right)^{40}\right] \Rightarrow a_1 = 3, r = \frac{3}{5}$

$$S_{41} = 5 + 3\left[\frac{1-\left(\frac{3}{5}\right)^{40}}{1-\left(\frac{3}{5}\right)}\right] = 5 + \frac{15}{2}\left[1-\left(\frac{3}{5}\right)^{40}\right] \approx 12.500$$

63. $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^n = \sum_{n=1}^{16} 2\left(\frac{4}{3}\right)^{n-1} = 2 + 2\left(\frac{4}{3}\right)^1 + 2\left(\frac{4}{3}\right)^2 + \cdots + 2\left(\frac{4}{3}\right)^{15} \Rightarrow a_1 = 2, r = \frac{4}{3}, n = 16$

$S_{16} = 2\left(\frac{1 - \left(\frac{4}{3}\right)^{16}}{1 - \frac{4}{3}}\right) \approx 592.647$

64. $\sum_{n=0}^{20} 10\left(\frac{1}{5}\right)^n = 10 + \sum_{n=1}^{20} 10\left(\frac{1}{5}\right)^n = 10 + \left[10\left(\frac{1}{5}\right)^1 + 10\left(\frac{1}{5}\right)^2 + 10\left(\frac{1}{5}\right)^3 + \cdots + 10\left(\frac{1}{5}\right)^{20}\right] \Rightarrow a_1 = 2, r = \frac{1}{5}$

$S_{21} = 10 + 2\left[\frac{1 - \left(\frac{1}{5}\right)^{20}}{1 - \left(\frac{1}{5}\right)}\right] = 10 + \frac{5}{2}\left[1 - \left(\frac{1}{5}\right)^{20}\right] \approx 12.500$

65. $\sum_{n=0}^{5} 300(1.06)^n = \sum_{n=1}^{6} 300(1.06)^{n-1}$

$= 300 + 300(1.06)^1 + 300(1.06)^2 + 300(1.06)^3 + 300(1.06)^4 + 300(1.06)^5 \Rightarrow a_1 = 300, \ r = 1.06$

$S_6 = 300\left[\frac{1 - (1.06)^6}{1 - 1.06}\right] \approx 2092.596$

66. $\sum_{n=0}^{6} 500(1.04)^n = 500 + \sum_{n=1}^{6} 500(1.04)^n = 500 + [500(1.04)^1 + 500(1.04)^2 + \cdots + 500(1.04)^6]$

$a_1 = 520, r = 1.04$

$S_7 = 500 + 520\left[\frac{1 - (1.04)^6}{1 - (1.04)}\right] = 500 - 13{,}000[1 - (1.04)^6] \approx 3949.147$

67. $\sum_{n=0}^{40} 2\left(-\frac{1}{4}\right)^n = 2 + 2\left(-\frac{1}{4}\right) + 2\left(-\frac{1}{4}\right)^2 + \cdots + 2\left(-\frac{1}{4}\right)^{40} \Rightarrow a_1 = 2, \ r = -\frac{1}{4}, n = 41$

$S_{41} = 2\left[\frac{1 - \left(-\frac{1}{4}\right)^{41}}{1 - \left(-\frac{1}{4}\right)}\right] = \frac{8}{5}\left[1 - \left(-\frac{1}{4}\right)^{41}\right] \approx 1.6 = \frac{8}{5}$

68. $\sum_{n=0}^{50} 10\left(\frac{2}{3}\right)^{n-1} = 15 + \sum_{n=1}^{50} 10\left(\frac{2}{3}\right)^{n-1} = 15 + \left[10 + 10\left(\frac{2}{3}\right)^1 + 10\left(\frac{2}{3}\right)^2 + \cdots + 10\left(\frac{2}{3}\right)^{49}\right] \Rightarrow a_1 = 10, r = \frac{2}{3}$

$S_{51} = 15 + 10\left[\frac{1 - \left(\frac{2}{3}\right)^{50}}{1 - \left(\frac{2}{3}\right)}\right] = 15 + 30\left[1 - \left(\frac{2}{3}\right)^{50}\right] \approx 45.000$

69. $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1} = 8 + 8\left(-\frac{1}{4}\right)^1 + 8\left(-\frac{1}{4}\right)^2 + \cdots + 8\left(-\frac{1}{4}\right)^9 \Rightarrow a_1 = 8, \ r = -\frac{1}{4}$

$S_{10} = 8\left[\frac{1 - \left(-\frac{1}{4}\right)^{10}}{1 - \left(-\frac{1}{4}\right)}\right] = \frac{32}{5}\left[1 - \left(-\frac{1}{4}\right)^{10}\right] \approx 6.400$

70. $\sum_{i=0}^{25} 8\left(-\frac{1}{2}\right)^i = 8 + \sum_{i=1}^{25} 8\left(-\frac{1}{2}\right)^i = 8 + \left[-4 + 8\left(-\frac{1}{2}\right)^2 + 8\left(-\frac{1}{2}\right)^3 + \cdots + 8\left(-\frac{1}{2}\right)^{25}\right] \Rightarrow a_1 = -4, r = -\frac{1}{2}$

$S_{26} = 8 - 4\left[\frac{1 - \left(-\frac{1}{2}\right)^{25}}{1 - \left(-\frac{1}{2}\right)}\right] = 8 - \frac{8}{3}\left[1 - \left(-\frac{1}{2}\right)^{25}\right] \approx 5.333$

71. $\sum_{i=1}^{10} 5\left(-\frac{1}{3}\right)^{i-1} = 5 + 5\left(-\frac{1}{3}\right)^1 + 5\left(-\frac{1}{3}\right)^2 + \cdots + 5\left(-\frac{1}{3}\right)^9 \Rightarrow a_1 = 5, r = -\frac{1}{3}, n = 10$

$S_{10} = 5\left(\frac{1 - \left(-\frac{1}{3}\right)^{10}}{1 - \left(-\frac{1}{3}\right)}\right) \approx 3.750$

72. $\sum_{i=1}^{100} 15\left(\frac{2}{3}\right)^{i-1} = 15 + 15\left(\frac{2}{3}\right)^1 + 15\left(\frac{2}{3}\right)^2 + \cdots + 15\left(\frac{2}{3}\right)^{99} \Rightarrow a_1 = 15, r = \frac{2}{3}$

$$S_{100} = 15\left[\frac{1 - \left(\frac{2}{3}\right)^{100}}{1 - \left(\frac{2}{3}\right)}\right] = 45\left[1 - \left(\frac{2}{3}\right)^{100}\right] \approx 45.000$$

73. $5 + 15 + 45 + \cdots + 3645$

$r = 3$ and $3645 = 5(3)^{n-1}$

$729 = 3^{n-1} \Rightarrow 6 = n - 1 \Rightarrow n = 7$

Thus, the sum can be written as $\sum_{n=1}^{7} 5(3)^{n-1}$.

74. $7 + 14 + 28 + \cdots + 896$

$a_1 = 7, r = 2$

$7(2)^{n-1} = 896$

$2^{n-1} = 128$

$2^{n-1} = 2^7$

$n - 1 = 7$

$n = 8$

Thus, the sum can be written as $\sum_{n=1}^{8} 7(2)^{n-1}$.

75. $2 - \frac{1}{2} + \frac{1}{8} - \cdots + \frac{1}{2048}$

$r = -\frac{1}{4}$ and $\frac{1}{2048} = 2\left(-\frac{1}{4}\right)^{n-1}$

By trial and error, we find that $n = 7$.

Thus, the sum can be written as $\sum_{n=1}^{7} 2\left(-\frac{1}{4}\right)^{n-1}$.

76. $15 - 3 + \frac{3}{5} - \cdots - \frac{3}{625}$

$a_1 = 15, r = -\frac{1}{5}$

$15\left(-\frac{1}{5}\right)^{n-1} = -\frac{3}{625}$

$\left(-\frac{1}{5}\right)^{n-1} = -\frac{1}{3125}$

$\left(-\frac{1}{5}\right)^{n} = \frac{1}{15{,}625}$

By trial and error, we find that $n = 6$.
Thus, the sum can be written as

$\sum_{n=1}^{6} 15\left(-\frac{1}{5}\right)^{n-1}$.

77. $0.1 + 0.4 + 1.6 + \cdots + 102.4$

$r = 4$ and $102.4 = 0.1(4)^{n-1}$

$1024 = 4^{n-1} \Rightarrow 5 = n - 1 \Rightarrow n = 6$

Thus, the sum can be written as $\sum_{n=1}^{6} 0.1(4)^{n-1}$.

78. $32 + 24 + 18 + \cdots + 10.125$

$a_1 = 32, r = \frac{3}{4}$

$32\left(\frac{3}{4}\right)^{n-1} = 10.125 = \frac{81}{8}$

$\left(\frac{3}{4}\right)^{n-1} = \frac{81}{256}$

$\left(\frac{3}{4}\right)^{n-1} = \left(\frac{3}{4}\right)^4$

$n - 1 = 4$

$n = 5$

Thus, the sum can be written as

$\sum_{n=1}^{5} 32\left(\frac{3}{4}\right)^{n-1}$.

79. $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots$

$a_1 = 1,\ r = \frac{1}{2}$

$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{a_1}{1-r} = \frac{1}{1-\left(\frac{1}{2}\right)} = 2$

80. $\sum_{n=0}^{\infty}2\left(\frac{2}{3}\right)^n = 2 + 2\left(\frac{2}{3}\right)^1 + 2\left(\frac{2}{3}\right)^2 + \cdots$

$a_1 = 2, r = \frac{2}{3}$

$\sum_{n=0}^{\infty}2\left(\frac{2}{3}\right)^n = \frac{a_1}{1-r} = \frac{2}{1-\frac{2}{3}} = 6$

81. $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = 1 + \left(-\frac{1}{2}\right)^1 + \left(-\frac{1}{2}\right)^2 + \cdots$

$a_1 = 1,\ r = -\frac{1}{2}$

$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \frac{a_1}{1-r} = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$

82. $\sum_{n=0}^{\infty}2\left(-\frac{2}{3}\right)^n = 2 + 2\left(-\frac{2}{3}\right)^1 + 2\left(-\frac{2}{3}\right)^2 + \cdots$

$a_1 = 2, r = -\frac{2}{3}$

$\sum_{n=0}^{\infty}2\left(-\frac{2}{3}\right)^n = \frac{a_1}{1-r} = \frac{2}{1-\left(-\frac{2}{3}\right)} = \frac{6}{5}$

83. $\sum_{n=0}^{\infty}4\left(\frac{1}{4}\right)^n = 4 + 4\left(\frac{1}{4}\right)^1 + 4\left(\frac{1}{4}\right)^2 + \cdots$

$a_1 = 4,\ r = \frac{1}{4}$

$\sum_{n=0}^{\infty}4\left(\frac{1}{4}\right)^n = \frac{a_1}{1-r} = \frac{4}{1-\left(\frac{1}{4}\right)} = \frac{16}{3}$

84. $\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n = 1 + \left(\frac{1}{10}\right)^1 + \left(\frac{1}{10}\right)^2 + \cdots$

$a_1 = 1, r = \frac{1}{10}$

$\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{10}} = \frac{10}{9}$

85. $\sum_{n=0}^{\infty}(0.4)^n = 1 + (0.4)^1 + (0.4)^2 + \cdots$

$a_1 = 1, r = 0.4$

$\sum_{n=0}^{\infty}(0.4)^n = \frac{1}{1-0.4} = \frac{5}{3}$

86. $\sum_{n=0}^{\infty}4(0.2)^n = 4 + 4(0.2)^1 + 4(0.2)^2 + \cdots$

$a_1 = 4, r = 0.2$

$\sum_{n=0}^{\infty}4(0.2)^n = \frac{4}{1-0.2} = 5$

87. $\sum_{n=0}^{\infty}-3(0.9)^n = -3 - 3(0.9)^1 - 3(0.9)^2 - \cdots$

$a_1 = -3, r = 0.9$

$\sum_{n=0}^{\infty}-3(0.9)^n = \frac{-3}{1-0.9} = -30$

88. $\sum_{n=0}^{\infty}[-10(0.2)^n] = -10 - 10(0.2)^1 - 10(0.2)^2 - \cdots$

$a_1 = -10, r = 0.2$

$\sum_{n=0}^{\infty}-10(0.2)^n = \frac{-10}{1-0.2} = -12.5$

89. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots = \sum_{n=0}^{\infty}8\left(\frac{3}{4}\right)^n = \frac{8}{1-\frac{3}{4}} = 32$

90. $9 + 6 + 4 + \frac{8}{3} + \cdots$

$a_1 = 9, r = \frac{2}{3}$

$\sum_{n=0}^{\infty}9\left(\frac{2}{3}\right)^n = \frac{9}{1-\frac{2}{3}} = 27$

91. $\frac{1}{9} - \frac{1}{3} + 1 - 3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{9}(-3)^n$

The sum is undefined because

$|r| = |-3| = 3 > 1.$

92. $\dfrac{-125}{36} + \dfrac{25}{6} - 5 + 6 - \cdots = \sum_{n=0}^{\infty} -\dfrac{125}{36}\left(-\dfrac{6}{5}\right)^n$

The sum is undefined because

$|r| = \left|-\dfrac{6}{5}\right| = \dfrac{6}{5} > 1.$

93. $0.\overline{36} = \sum_{n=0}^{\infty} 0.36(0.01)^n = \dfrac{0.36}{1-0.01} = \dfrac{0.36}{0.99} = \dfrac{36}{99} = \dfrac{4}{11}$

94. $0.\overline{297} = \sum_{n=0}^{\infty} 0.297(0.001)^n = \dfrac{0.297}{1-0.001} = \dfrac{0.297}{0.999}$

$= \dfrac{297}{999} = \dfrac{11}{37}$

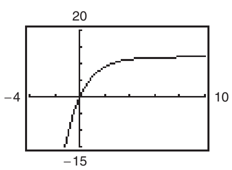
95. $0.3\overline{18} = 0.3 + \sum_{n=0}^{\infty} 0.018(0.01)^n = \dfrac{3}{10} + \dfrac{0.018}{1-0.01}$

$= \dfrac{3}{10} + \dfrac{0.018}{0.99} = \dfrac{3}{10} + \dfrac{18}{990} = \dfrac{3}{10} + \dfrac{2}{110}$

$= \dfrac{35}{110} = \dfrac{7}{22}$

96. $1.3\overline{8} = 1.3 + \sum_{n=0}^{\infty} 0.08(0.1)^n = 1.3 + \dfrac{0.08}{1-0.1} = 1.3 + \dfrac{0.08}{0.9} = 1\dfrac{3}{10} + \dfrac{4}{45} = 1\dfrac{7}{18} = \dfrac{25}{18}$

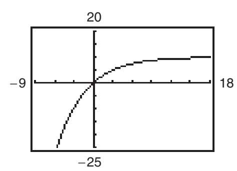
97. $f(x) = 6\left[\dfrac{1-(0.5)^x}{1-(0.5)}\right], \sum_{n=0}^{\infty} 6\left(\dfrac{1}{2}\right)^n = \dfrac{6}{1-\frac{1}{2}} = 12$



The horizontal asymptote of $f(x)$ is $y = 12$. This corresponds to the sum of the series.

98. $f(x) = 2\left[\dfrac{1-(0.8)^x}{1-(0.8)}\right], \sum_{n=0}^{\infty} 2\left(\dfrac{4}{5}\right)^n = \dfrac{2}{1-\frac{4}{5}} = 10$

The horizontal asymptote of $f(x)$ is $y = 10$. This corresponds to the sum of the series.



99. (a) $a_n \approx 1190.88(1.006)^n$

(b) The population is growing at a rate of 0.6% per year.

(c) For 2010, let $n = 20$: $a_n = 1190.88(1.006)^{20}$

≈ 1342.2 million

(d) $1190.88(1.006)^n = 1320$

$$1.006^n = \frac{1320}{1190.88}$$

$$\ln 1.006^n = \ln\left(\frac{1320}{1190.88}\right)$$

$$n \ln 1.006 = \ln\left(\frac{1320}{1190.88}\right)$$

$$n = \frac{\ln\left(\frac{1320}{1190.88}\right)}{\ln 1.006} \approx 17.21$$

This corresponds with the year 2008.

100. $A = P\left(1 + \dfrac{r}{n}\right)^{nt} = 1000\left(1 + \dfrac{0.06}{n}\right)^{n(10)}$

(a) $n = 1,\ A = 1000(1 + 0.06)^{10} \approx \$1790.85$

(b) $n = 2,\ A = 1000\left(1 + \dfrac{0.06}{2}\right)^{2(10)} \approx \$1806.11$

(c) $n = 4,\ A = 1000\left(1 + \dfrac{0.06}{4}\right)^{4(10)} \approx \$1814.02$

(d) $n = 12,\ A = 1000\left(1 + \dfrac{0.06}{12}\right)^{12(10)} \approx \$1819.40$

(e) $n = 365,\ A = 1000\left(1 + \dfrac{0.06}{365}\right)^{365(10)} \approx \$1822.03$

101. $A = P\left(1 + \frac{r}{n}\right)^{nt} = 2500\left(1 + \frac{0.02}{n}\right)^{n(20)}$

(a) $n = 1$: $A = 2500\left(1 + \frac{0.02}{1}\right)^{(1)(20)} \approx \$3714.87$

(b) $n = 2$: $A = 2500\left(1 + \frac{0.02}{2}\right)^{(2)(20)} \approx \$3722.16$

(c) $n = 4$: $A = 2500\left(1 + \frac{0.02}{4}\right)^{(4)(20)} \approx \$3725.85$

(d) $n = 12$: $A = 2500\left(1 + \frac{0.02}{12}\right)^{(12)(20)} \approx \$3728.32$

(e) $n = 365$: $A = 2500\left(1 + \frac{0.02}{365}\right)^{(365)(20)} \approx \$3729.52$

102. $V_5 = 135{,}000(0.70)^5 = \$22{,}689.45$

103. $A = \sum_{n=1}^{60} 100\left(1 + \frac{0.06}{12}\right)^n = \sum_{n=1}^{60} 100(1.005)^n = 100(1.005) \cdot \frac{[1 - 1.005^{60}]}{[1 - 1.005]} \approx \$7011.89$

104. $A = \sum_{n=1}^{60} 50\left(1 + \frac{0.08}{12}\right)^n$

$= 50(1.006666667)\left(\frac{1 - (1.006666667)^{60}}{1 - 1.006666667}\right)$

$\approx \$3698.34$

105. Let $N = 12t$ be the total number of deposits.

$$A = P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^2 + \cdots + P\left(1 + \frac{r}{12}\right)^N$$

$$= \left(1 + \frac{r}{12}\right)\left[P + P\left(1 + \frac{r}{12}\right) + \cdots + P\left(1 + \frac{r}{12}\right)^{N-1}\right]$$

$$= P\left(1 + \frac{r}{12}\right)\sum_{n=1}^{N}\left(1 + \frac{r}{12}\right)^{n-1}$$

$$= P\left(1 + \frac{r}{12}\right)\left[\frac{1 - \left(1 + \frac{r}{12}\right)^N}{1 - \left(1 + \frac{r}{12}\right)}\right]$$

$$= P\left(1 + \frac{r}{12}\right)\left(-\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^N\right]$$

$$= P\left(\frac{12}{r} + 1\right)\left[-1 + \left(1 + \frac{r}{12}\right)^N\right]$$

$$= P\left[\left(1 + \frac{r}{12}\right)^N - 1\right]\left(1 + \frac{12}{r}\right)$$

$$= P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right)$$

106. Let $N = 12t$ be the total number of deposits.

$$A = Pe^{r/12} + Pe^{2r/12} + \cdots + Pe^{Nr/12}$$

$$= \sum_{n=1}^{N} Pe^{r/12 \cdot n}$$

$$= Pe^{r/12}\frac{(1 - (e^{r/12})^N)}{(1 - e^{r/12})}$$

$$= Pe^{r/12}\frac{(1 - (e^{r/12})^{12t})}{1 - e^{r/12}}$$

$$= \frac{Pe^{r/12}(e^{rt} - 1)}{(e^{r/12} - 1)}$$

107. $P = \$50$, $r = 7\%$, $t = 20$ years

(a) Compounded monthly:

$$A = 50\left[\left(1 + \frac{0.07}{12}\right)^{12(20)} - 1\right]\left(1 + \frac{12}{0.07}\right)$$

$\approx \$26{,}198.27$

(b) Compounded continuously:

$$A = \frac{50e^{0.07/12}(e^{0.07(20)} - 1)}{e^{0.07/12} - 1} \approx \$26{,}263.88$$

108. $P = \$75,\ r = 3\%,\ t = 25$ years

(a) Compounded monthly: $A = 75\left[\left(1 + \frac{0.03}{12}\right)^{12(25)} - 1\right]\left(1 + \frac{12}{0.03}\right) \approx \$33{,}534.21$

(b) Compounded continuously: $A = \dfrac{75e^{0.03/12}(e^{0.03(25)} - 1)}{e^{0.03/12} - 1} \approx \$33{,}551.91$

109. $P = \$100,\ r = 10\%,\ t = 40$ years

(a) Compounded monthly: $A = 100\left[\left(1 + \frac{0.10}{12}\right)^{12(40)} - 1\right]\left(1 + \frac{12}{0.10}\right) \approx \$637{,}678.02$

(b) Compounded continuously: $A = \dfrac{100e^{0.10/12}(e^{(0.10)(40)} - 1)}{e^{0.10/12} - 1} \approx \$645{,}861.43$

110. $P = \$20,\ r = 6\%,\ t = 50$ years

(a) Compounded monthly: $A = 20\left[\left(1 + \frac{0.06}{12}\right)^{12(50)} - 1\right]\left(1 + \frac{12}{0.06}\right) \approx \$76{,}122.54$

(b) Compounded continuously: $A = \dfrac{20e^{0.06/12}(e^{0.06(50)} - 1)}{e^{0.06/12} - 1} \approx \$76{,}533.16$

111.
$$P = W\sum_{n=1}^{12t}\left[\left(1 + \frac{r}{12}\right)^{-1}\right]^n$$
$$= W\left(1 + \frac{r}{12}\right)^{-1}\left[\frac{1 - \left(1 + \frac{r}{12}\right)^{-12t}}{1 - \left(1 + \frac{r}{12}\right)^{-1}}\right]$$
$$= W\left(\frac{1}{1 + \frac{r}{12}}\right)\frac{\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right]}{1 - \frac{1}{\left(1 + \frac{r}{12}\right)}}$$
$$= W\frac{\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right]}{\left(1 + \frac{r}{12}\right) - 1}$$
$$= W\left(\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right]$$

112. $W = \$2000,\ t = 20,\ r = 9\%$
$$P = W\left(\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right]$$
$$P = 2000\left(\frac{12}{0.09}\right)\left[1 - \left(1 + \frac{0.09}{12}\right)^{-12(20)}\right] \approx \$222{,}289.91$$

113. $\displaystyle\sum_{n=1}^{\infty} 400(0.75)^n = \frac{300}{1 - 0.75} = \$1200$

114. $a_1 = 250(0.80) = 200$

$r = 80\% = 0.80$

$$\text{Amount put back into economy} = \sum_{n=1}^{\infty} 250(0.80)^n = \frac{200}{1 - 0.80} = \frac{200}{0.20} = \$1000$$

115. $\sum_{n=1}^{\infty} 600(0.725)^n = \dfrac{435}{1 - 0.725} \approx \$1581.82$

116. $a_1 = 450(0.775) = 348.75$

$r = 77.5\% = 0.775$

$$\text{Amount put back into economy} = \sum_{n=1}^{\infty} 450(0.775)^n = \frac{348.75}{1 - 0.775} = \frac{348.75}{0.225} = \$1550$$

117. $64 + 32 + 16 + 8 + 4 + 2 = 126$

Total area of shaded region is approximately 126 square inches.

118. $a_n = 54.6e^{0.172n}, n = 4, 5, \ldots, 13$

$r = e^{0.172n}$

$a_1 = 54.6e^{0.172} = 64.84721$

$S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\dfrac{1 - r^n}{1 - r}\right)$

$S = S_{13} - S_3$

$= 64.84721\left(\dfrac{1 - e^{(0.172)(13)}}{1 - e^{0.172}}\right) - 64.84721\left(\dfrac{1 - e^{(0.172)(3)}}{1 - e^{0.172}}\right)$

$= 2887.141484 - 233.336893 = 2653.80$

The total sales over the 10-year period is \$2653.80 million.

119. $a_n = 30{,}000(1.05)^{n-1}$

$$T = \sum_{n=1}^{40} 30{,}000(1.05)^{n-1} = 30{,}000\frac{(1 - 1.05^{40})}{(1 - 1.05)} \approx \$3{,}623{,}993.23$$

120. (a) Total distance $= \left[\sum_{n=0}^{\infty} 32(0.81)^n\right] - 16 = \dfrac{32}{1 - 0.81} - 16 \approx 152.42$ feet

(b) $t = 1 + 2\sum_{n=1}^{\infty} (0.9)^n = 1 + 2\left[\dfrac{0.9}{1 - 0.9}\right] = 19$ seconds

121. False. A sequence is geometric if the ratios of consecutive terms are the same.

122. False. $a_n = a_1 r^{n-1}$, NOT $r a_1^{\,n-1}$

The nth-term of a geometric sequence can be found by multiplying its first term by its common ratio raised to the $(n - 1)$th power.

123. Given a real number r between -1 and 1, as the exponent n increases, r^n approaches zero.

124. *Sample answer:*

$$\sum_{n=1}^{199} 4(-1)^{n-1} \text{ and } \sum_{n=1}^{8} -\frac{4}{85}(-2)^{n-1}$$

125. $g(x) = x^2 - 1$

$g(x + 1) = (x + 1)^2 - 1$

$= x^2 + 2x + 1 - 1 = x^2 + 2x$

126. $f(x) = 3x + 1$

$f(x + 1) = 3(x + 1) + 1 = 3x + 4$

127. $f(x) = 3x + 1, g(x) = x^2 - 1$

$$\begin{aligned} f(g(x+1)) &= f(x^2 + 2x) \\ &= 3(x^2 + 2x) + 1 \\ &= 3x^2 + 6x + 1 \end{aligned}$$

128. $g(x) = x^2 - 1$

$$\begin{aligned} g(f(x+1)) &= g(3x + 4) \quad \text{From Exercise 126} \\ &= (3x+4)^2 - 1 \\ &= 9x^2 + 24x + 15 \end{aligned}$$

129. $9x^3 - 64x = x(9x^2 - 64) = x(3x + 8)(3x - 8)$

130. $x^2 + 4x - 63$ Does not factor

131. $6x^2 - 13x - 5 = (3x + 1)(2x - 5)$

132. $16x^2 - 4x^4 = 4x^2(4 - x^2)$

$$= 4x^2(2 + x)(2 - x)$$

133. $\dfrac{3}{x+3} \cdot \dfrac{x(x+3)}{x-3} = \dfrac{3x}{x-3}, x \neq -3$

134. $\dfrac{\cancel{x-2}}{\cancel{x+7}} \cdot \dfrac{\overset{1}{\cancel{2x}}\cancel{(x+7)}}{\underset{3}{\cancel{6x}}\cancel{(x-2)}} = \dfrac{1}{3}, x \neq -7, 2$

135. $\dfrac{x}{3} \div \dfrac{3x}{6x+3} = \dfrac{x}{3} \cdot \dfrac{3(2x+1)}{3x} = \dfrac{2x+1}{3}, x \neq 0, -\dfrac{1}{2}$

136. $\dfrac{x-5}{x-3} \div \dfrac{10-2x}{2(3-x)} = \dfrac{\cancel{x-5}}{\cancel{x-3}} \cdot \dfrac{\cancel{-2}\cancel{(x-3)}}{\cancel{-2}\cancel{(x-5)}} = 1, x \neq 3, 5$

137.
$$\begin{aligned} 5 + \frac{7}{x+2} + \frac{2}{x-2} &= \frac{5(x+2)(x-2) + 7(x-2) + 2(x+2)}{(x+2)(x-2)} \\ &= \frac{5(x^2-4) + 7(x-2) + 2(x+2)}{(x+2)(x-2)} \\ &= \frac{5x^2 - 20 + 7x - 14 + 2x + 4}{(x+2)(x-2)} = \frac{5x^2 + 9x - 30}{(x+2)(x-2)} \end{aligned}$$

138.
$$\begin{aligned} 8 - \frac{x-1}{x+4} - \frac{4}{x-1} - \frac{x+4}{(x-1)(x+4)} &= \frac{8(x-1)(x+4) - (x-1)^2 - 4(x+4) - (x+4)}{(x-1)(x+4)} \\ &= \frac{8(x^2 + 3x - 4) - (x^2 - 2x + 1) - 4x - 16 - x - 4}{(x-1)(x+4)} \\ &= \frac{8x^2 + 24x - 32 - x^2 + 2x - 1 - 4x - 16 - x - 4}{(x-1)(x+4)} \\ &= \frac{7x^2 + 21x - 53}{(x-1)(x+4)} \end{aligned}$$

139. Answers will vary.

Section 9.4 Mathematical Induction

- You should be sure that you understand the principle of mathematical induction. If P_n is a statement involving the positive integer n, where P_1 is true and the truth of P_k implies the truth of P_{k+1} for every positive k, then P_n is true for all positive integers n.
- You should be able to verify (by induction) the formulas for the sums of powers of integers and be able to use these formulas.
- You should be able to calculate the first and second differences of a sequence.
- You should be able to find the quadratic model for a sequence, when it exists.

Vocabulary Check

1. mathematical induction
2. first
3. arithmetic
4. second

1. $P_k = \dfrac{5}{k(k+1)}$

$P_{k+1} = \dfrac{5}{(k+1)[(k+1)+1]} = \dfrac{5}{(k+1)(k+2)}$

2. $P_k = \dfrac{1}{2(k+2)}$

$P_{k+1} = \dfrac{1}{2(k+1+2)} = \dfrac{1}{2(k+3)}$

3. $P_k = \dfrac{k^2(k+1)^2}{4}$

$P_{k+1} = \dfrac{(k+1)^2[(k+1)+1]^2}{4} = \dfrac{(k+1)^2(k+2)^2}{4}$

4. $P_k = \dfrac{k}{3}(2k+1)$

$P_{k+1} = \dfrac{k+1}{3}[2(k+1)+1] = \dfrac{k+1}{3}(2k+3)$

5. 1. When $n = 1$, $S_1 = 2 = 1(1+1)$.

2. Assume that

$S_k = 2 + 4 + 6 + 8 + \cdots + 2k = k(k+1)$.

Then,

$S_{k+1} = 2 + 4 + 6 + 8 + \cdots + 2k + 2(k+1)$

$= S_k + 2(k+1) = k(k+1) + 2(k+1) = (k+1)(k+2)$.

Therefore, we conclude that the formula is valid for all positive integer values of n.

6. 1. When $n = 1$, $S_1 = 3 = 1(2 \cdot 1 + 1)$.

2. Assume that

$S_k = 3 + 7 + 11 + 15 + \cdots + (4k-1) = k(2k+1)$.

Then,

$S_{k+1} = S_k + a_{k+1} = (3 + 7 + 11 + 15 + \cdots + (4k-1)) + [4(k+1) - 1]$

$= k(2k+1) + (4k+3)$

$= 2k^2 + 5k + 3$

$= (k+1)(2k+3)$

$= (k+1)[2(k+1)+1]$.

Therefore, we conclude that this formula is valid.

7. 1. When $n = 1$, $S_1 = 2 = \frac{1}{2}(5(1) - 1)$.

2. Assume that

$$S_k = 2 + 7 + 12 + 17 + \cdots + (5k - 3) = \frac{k}{2}(5k - 1).$$

Then,

$$\begin{aligned} S_{k+1} &= 2 + 7 + 12 + 17 + \cdots + (5k - 3) + [5(k + 1) - 3] \\ &= S_k + (5k + 5 - 3) = \frac{k}{2}(5k - 1) + 5k + 2 \\ &= \frac{5k^2 - k + 10k + 4}{2} = \frac{5k^2 + 9k + 4}{2} \\ &= \frac{(k + 1)(5k + 4)}{2} = \frac{(k + 1)}{2}[5(k + 1) - 1]. \end{aligned}$$

Therefore, we conclude that this formula is valid for all positive integer values of n.

8. 1. When $n = 1$,

$$S_1 = 1 = \frac{1}{2}(3 \cdot 1 - 1).$$

2. Assume that

$$S_k = 1 + 4 + 7 + 10 + \cdots + (3k - 2) = \frac{k}{2}(3k - 1).$$

Then,

$$\begin{aligned} S_{k+1} &= S_k + a_{k+1} = (1 + 4 + 7 + 10 + \cdots + (3k - 2)) + (3(k + 1) - 2) \\ &= \frac{k}{2}(3k - 1) + (3k + 1) \\ &= \frac{3k^2 - k + 6k + 2}{2} \\ &= \frac{3k^2 + 5k + 2}{2} \\ &= \frac{(k + 1)(3k + 2)}{2} \\ &= \frac{k + 1}{2}[3(k + 1) - 1]. \end{aligned}$$

Therefore, we conclude that this formula is valid for all positive integer values of n.

9. 1. When $n = 1$, $S_1 = 1 = 2^1 - 1$.

2. Assume that

$$S_k = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} = 2^k - 1.$$

Then,

$$\begin{aligned} S_{k+1} &= 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} + 2^k \\ &= S_k + 2^k = 2^k - 1 + 2^k = 2(2^k) - 1 = 2^{k+1} - 1. \end{aligned}$$

Therefore, we conclude that this formula is valid for all positive integer values of n.

10. 1. When $n = 1$, $S_1 = 2 = 3^1 - 1$.

2. Assume that

$$S_k = 2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1}) = 3^k - 1.$$

Then,

$$\begin{aligned} S_{k+1} &= S_k + a_{k+1} \\ &= [2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1})] + 2 \cdot 3^{k+1-1} \\ &= 3^k - 1 + 2 \cdot 3^k \\ &= 3 \cdot 3^k - 1 \\ &= 3^{k+1} - 1. \end{aligned}$$

Therefore, we conclude that this formula is valid for all positive integer values of n.

11. 1. When $n = 1$, $S_1 = 1 = \dfrac{1(1 + 1)}{2}$.

2. Assume that

$$S_k = 1 + 2 + 3 + 4 + \cdots + k = \frac{k(k + 1)}{2}.$$

Then,

$$\begin{aligned} S_{k+1} &= 1 + 2 + 3 + 4 + \cdots + k + (k + 1) \\ &= S_k + (k + 1) = \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}. \end{aligned}$$

Therefore, we conclude that this formula is valid for all positive integer values of n.

12. $S_n = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$.

1. When $n = 1$, $S_n = 1^3 = 1 = \dfrac{1^2(1 + 1)^2}{4}$.

2. Assume that

$$S_k = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \frac{k^2(k + 1)^2}{4}.$$

Then,

$$\begin{aligned} S_{k+1} &= 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k + 1)^3 \\ &= S_k + (k + 1)^3 = \frac{k^2(k + 1)^2}{4} + (k + 1)^3 = \frac{k^2(k + 1)^2 + 4(k + 1)^3}{4} \\ &= \frac{(k + 1)^2(k^2 + 4k + 4)}{4} = \frac{(k + 1)^2(k + 2)^2}{4} = \frac{(k + 1)^2[(k + 1) + 1]^2}{4} \end{aligned}$$

Therefore, we conclude that this formula is valid for all positive integer values of n.

13. 1. When $n = 1$, $S_1 = 1 = \dfrac{(1)^2(1+1)^2(2(1)^2 + 2(1) - 1)}{12}$.

2. Assume that

$$S_k = \sum_{i=1}^{k} i^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12}.$$

Then,

$$\begin{aligned}
S_{k+1} = \sum_{i=1}^{k+1} i^5 &= \left(\sum_{i=1}^{k} i^5\right) + (k+1)^5 \\
&= \frac{k^2(k+1)^2(2k^2+2k-1)}{12} + \frac{12(k+1)^5}{12} \\
&= \frac{(k+1)^2[k^2(2k^2+2k-1) + 12(k+1)^3]}{12} \\
&= \frac{(k+1)^2[2k^4 + 2k^3 - k^2 + 12(k^3 + 3k^2 + 3k + 1)]}{12} \\
&= \frac{(k+1)^2[2k^4 + 14k^3 + 35k^2 + 36k + 12]}{12} \\
&= \frac{(k+1)^2(k^2+4k+4)(2k^2+6k+3)}{12} \\
&= \frac{(k+1)^2(k+2)^2[2(k+1)^2 + 2(k+1) - 1]}{12}.
\end{aligned}$$

Therefore, we conclude that this formula is valid for all positive integer values of n.

Note: The easiest way to complete the last two steps is to "work backwards." Start with the desired expression for S_{k+1} and multiply out to show that it is equal to the expression you found for $S_k + (k+1)^5$.

14. 1. When $n = 1$,

$$S_1 = 1^4 = \frac{1(1+1)(2\cdot 1+1)(3\cdot 1^2 + 3\cdot 1 - 1)}{30}.$$

2. Assume that

$$S_k = \sum_{i=1}^{k} i^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}.$$

Then,

$$\begin{aligned}
S_{k+1} = S_k + a_{k+1} &= S_k + (k+1)^4 \\
&= \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4 \\
&= \frac{k(k+1)(2k+1)(3k^2+3k-1) + 30(k+1)^4}{30} \\
&= \frac{(k+1)[k(2k+1)(3k^2+3k-1) + 30(k+1)^3]}{30} \\
&= \frac{(k+1)(6k^4 + 39k^3 + 91k^2 + 89k + 30)}{30} \\
&= \frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30} \\
&= \frac{(k+1)(k+2)(2(k+1)+1)(3(k+1)^2 + 3(k+1) - 1)}{30}.
\end{aligned}$$

Therefore, we conclude that this formula is valid for all positive integer values of n.

15. 1. When $n = 1, S_1 = 2 = \dfrac{1(2)(3)}{3}$.

2. Assume that

$$S_k = 1(2) + 2(3) + 3(4) + \cdot \cdot \cdot + k(k + 1) = \frac{k(k + 1)(k + 2)}{3}.$$

Then,

$$\begin{aligned} S_{k+1} &= 1(2) + 2(3) + 3(4) + \cdot \cdot \cdot + k(k + 1) + (k + 1)(k + 2) \\ &= S_k + (k + 1)(k + 2) = \frac{k(k + 1)(k + 2)}{3} + \frac{3(k + 1)(k + 2)}{3} \\ &= \frac{(k + 1)(k + 2)(k + 3)}{3}. \end{aligned}$$

Therefore, we conclude that this formula is valid for all positive integer values of n.

16. 1. When $n = 1$,

$$S_1 = \frac{1}{3} = \frac{1}{2 \cdot 1 + 1}.$$

2. Assume that

$$S_k = \sum_{i=1}^{k} \frac{1}{(2i - 1)(2i + 1)} = \frac{k}{2k + 1}.$$

Then,

$$\begin{aligned} S_{k+1} &= S_k + a_{k+1} = S_k + \frac{1}{(2(k + 1) - 1)(2(k + 1) + 1)} \\ &= \frac{k}{2k + 1} + \frac{1}{(2k + 1)(2k + 3)} \\ &= \frac{k(2k + 3) + 1}{(2k + 1)(2k + 3)} \\ &= \frac{2k^2 + 3k + 1}{(2k + 1)(2k + 3)} \\ &= \frac{(2k + 1)(k + 1)}{(2k + 1)(2k + 3)} \\ &= \frac{k + 1}{2(k + 1) + 1}. \end{aligned}$$

Therefore, we conclude that this formula is valid for all positive integer values of n.

17. 1. When $n = 4$, $4! = 24$ and $2^4 = 16$, thus $4! > 2^4$.

2. Assume

$k! > 2^k, \; k > 4$.

Then,

$(k + 1)! = k!(k + 1) > 2^k(2)$ since $k! > 2^k$ and $k + 1 > 2$.

Thus, $(k + 1)! > 2^{k+1}$.

Therefore, by extended mathematical induction, the inequality is valid for all integers n such that $n \geq 4$.

18. 1. When $n = 7$, $\left(\frac{4}{3}\right)^7 \approx 7.4915 > 7$.

2. Assume that $\left(\frac{4}{3}\right)^k > k, k > 7$.

Then, $\left(\frac{4}{3}\right)^{k+1} = \left(\frac{4}{3}\right)^k\left(\frac{4}{3}\right) > k\left(\frac{4}{3}\right) = k + \frac{k}{3} > k + 1$ for $k > 7$.

Thus, $\left(\frac{4}{3}\right)^{k+1} > k + 1$.

Therefore, the inequality $\left(\frac{4}{3}\right)^n > n$ is valid for all integers n such that $n \geq 7$.

19. 1. When $n = 2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \approx 1.707$ and $\sqrt{2} \approx 1.414$, thus $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$.

2. Assume that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k}, k > 2.$$

Then,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

Now it is sufficient to show that

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1},\ k > 2,$$

or equivalently (multiplying by $\sqrt{k+1}$),

$$\sqrt{k}\,\sqrt{k+1} + 1 > k + 1.$$

This is true because

$$\sqrt{k}\,\sqrt{k+1} + 1 > \sqrt{k}\,\sqrt{k} + 1 = k + 1.$$

Therefore,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}.$$

Therefore, by extended mathematical induction, the inequality is valid for all integers n such that $n \geq 2$.

20. 1. When $n = 1$, $\left(\frac{x}{y}\right)^2 < \left(\frac{x}{y}\right)$ and $(0 < x < y)$.

2. Assume that

$$\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)^k$$

$$\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)^k \Rightarrow \left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^k \Rightarrow \left(\frac{x}{y}\right)^{k+2} < \left(\frac{x}{y}\right)^{k+1}.$$

Therefore, $\left(\frac{x}{y}\right)^{n+1} < \left(\frac{x}{y}\right)^n$ for all integers $n \geq 1$.

21. $(1 + a)^n \geq na$, $n \geq 1$ and $a > 0$

Since a is positive, then all of the terms in the binomial expansion are positive.

$(1 + a)^n = 1 + na + \cdots + na^{n-1} + a^n > na$

22. $2n^2 > (n+1)^2, n \geq 3$

1. For $n = 3$, the statement is true, because $2(3)^2 = 18 > (3+1)^2 = 16$.
2. Assuming that $2k^2 > (k+1)^2$ you need to show that $2(k+1)^2 > (k+2)^2$. For $n = k$, you have

$$\begin{aligned}(k+2)^2 &= k^2 + 4k + 4\\ &= k^2 + 2k + 1 + 2k + 3\\ &= (k+1)^2 + 2k + 3.\end{aligned}$$

By the assumption $(k+1)^2 < 2k^2$, you have

$$(k+1)^2 + 2k + 3 < 2k^2 + 2k + 3.$$

Because $2k + 3 < 4k + 2$, or $1 < 2k$ for all $k > 3$, you can say that

$$2k^2 + 2k + 3 < 2k^2 + 4k + 2 = 2(k+1)^2.$$

It follows that $(k+2)^2 < 2k^2 + 2k + 3 < 2(k+1)^2$

or $2(k+1)^2 > (k+2)^2$.

Therefore, $2n^2(n+1)^2$ for all $n \geq 3$.

23. 1. When $n = 1$, $(ab)^1 = a^1b^1 = ab$.

2. Assume that $(ab)^k = a^kb^k$.

$$\text{Then, } (ab)^{k+1} = (ab)^k(ab) = a^kb^kab = a^{k+1}b^{k+1}.$$

Thus, $(ab)^n = a^nb^n$.

24. 1. When $n = 1$, $\left(\frac{a}{b}\right)^1 = \frac{a^1}{b^1}$.

2. Assume that $\left(\frac{a}{b}\right)^k = \frac{a^k}{b^k}$.

$$\text{Then, } \left(\frac{a}{b}\right)^{k+1} = \left(\frac{a}{b}\right)^k\left(\frac{a}{b}\right) = \frac{a^k}{b^k}\cdot\frac{a}{b} = \frac{a^{k+1}}{b^{k+1}}.$$

Thus, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

25. 1. When $n = 2$, $(x_1x_2)^{-1} = \frac{1}{x_1x_2} = \frac{1}{x_1}\cdot\frac{1}{x_2} = x_1^{-1}x_2^{-1}$.

2. Assume that

$$(x_1x_2x_3 \cdots x_k)^{-1} = x_1^{-1}x_2^{-1}x_3^{-1} \cdots x_k^{-1}.$$

Then,

$$\begin{aligned}(x_1x_2x_3 \cdots x_kx_{k+1})^{-1} &= [(x_1x_2x_3 \cdots x_k)x_{k+1}]^{-1}\\ &= (x_1x_2x_3 \ldots x_k)^{-1}x_{k+1}^{-1}\\ &= x_1^{-1}x_2^{-1}x_3^{-1} \cdots x_k^{-1}x_{k+1}^{-1}.\end{aligned}$$

Thus, the formula is valid.

26. 1. When $n = 1$, $\ln x_1 = \ln x_1$.

2. Assume that $\ln(x_1x_2x_3 \ldots x_k) = \ln x_1 + \ln x_2 + \ln x_3 + \cdots + \ln x_k$.

$$\begin{aligned}\text{Then, } \ln(x_1x_2x_3 \ldots x_k\,x_{k+1}) &= \ln[(x_1x_2x_3 \ldots x_k)x_{k+1}]\\ &= \ln(x_1x_2x_3 \ldots x_k) + \ln x_{k+1}\\ &= \ln x_1 + \ln x_2 + \ln x_3 + \cdots + \ln x_k + \ln x_{k+1}.\end{aligned}$$

Thus, $\ln(x_1x_2x_3 \ldots x_n) = \ln x_1 + \ln x_2 + \ln x_3 + \cdots \ln x_n$.

27. 1. When $n = 1$, $x(y_1) = xy_1$.

2. Assume that

$$x(y_1 + y_2 + \cdots + y_k) = xy_1 + xy_2 + \cdots + xy_k.$$

Then,

$$\begin{aligned}xy_1 + xy_2 + \cdots + xy_k + xy_{k+1} &= x(y_1 + y_2 + \cdots + y_k) + xy_{k+1}\\ &= x[(y_1 + y_2 + \cdots + y_k) + y_{k+1}]\\ &= x(y_1 + y_2 + \cdots + y_k + y_{k+1}).\end{aligned}$$

Hence, the formula holds.

28. 1. When $n = 1$, $a + bi$ and $a - bi$ are complex conjugates by definition.

2. Assume that $(a + bi)^k$ and $(a - bi)^k$ are complex conjugates. That is, if $(a + bi)^k = c + di$, then $(a - bi)^k = c - di$.

Then,

$$\begin{aligned}(a + bi)^{k+1} &= (a + bi)^k(a + bi) = (c + di)(a + bi)\\ &= (ac - bd) + i(bc + ad)\end{aligned}$$

and $$\begin{aligned}(a - bi)^{k+1} &= (a - bi)^k\,(a - bi) = (c - di)(a - bi)\\ &= (ac - bd) - i(bc + ad).\end{aligned}$$

This implies that $(a + bi)^{k+1}$ and $(a - bi)^{k+1}$ are complex conjugates.

Therefore, $(a + bi)^n$ and $(a - bi)^n$ are complex conjugates for $n \geq 1$.

29. 1. When $n = 1$, $[1^3 + 3(1)^2 + 2(1)] = 6$ and 3 is a factor.

2. Assume that 3 is a factor of $k^3 + 3k^2 + 2k$.

Then,

$$\begin{aligned}(k + 1)^3 + 3(k + 1)^2 + 2(k + 1) &= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2\\ &= (k^3 + 3k^2 + 2k) + (3k^2 + 9k + 6)\\ &= (k^3 + 3k^2 + 2k) + 3(k^2 + 3k + 2).\end{aligned}$$

Since 3 is a factor of $(k^3 + 3k^2 + 2k)$, our assumption, and 3 is a factor of $3(k^2 + 3k + 2)$, we conclude that 3 is a factor of the whole sum.

Thus, 3 is a factor of $(n^3 + 3n^2 + 2n)$ for every positive integer n.

30. Prove 3 is a factor of $n^3 - n + 3$ for all positive integers n.

1. When $n = 1, 1^3 - 1 + 3 = 3$ and 3 is a factor.

2. Assume that 3 is a factor of $k^3 - k + 3$.

Then,

$$\begin{aligned}(k + 1)^3 - (k + 1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3\\ &= k^3 + 3k^2 + 2k + 3\\ &= (k^3 - k + 3) + 3k^2 + 3k\\ &= (k^3 - k + 3) + 3k(k + 1).\end{aligned}$$

Since 3 is a factor of each term, 3 is a factor of the sum.

Thus, 3 is a factor of $n^3 - n + 3$ for all positive integers n.

31. A factor of $n^4 - n + 4$ is 2.

1. When $n = 1$, $1^4 - 1 + 4 = 4$ and 2 is a factor.

2. Assume that 2 is a factor of $k^4 - k + 4$.

Then,

$$\begin{aligned}(k + 1)^4 - (k + 1) + 4 &= k^4 + 4k^3 + 6k^2 + 4k + 1 - k - 1 + 4\\ &= (k^4 - k + 4) + (4k^3 + 6k^2 + 4k)\\ &= (k^4 - k + 4) + 2(2k^3 + 3k^2 + 2k).\end{aligned}$$

Since 2 is a factor of $k^4 - k + 4$, our assumption, and 2 is a factor of $2(2k^3 + 3k^2 + 2k)$, we conclude that 2 is a factor of the entire expression.

Thus, 2 is a factor of $n^4 - n + 4$ for every positive integer n.

32. Prove 3 is a factor of $2^{2n+1} + 1$ for all positive integers n.

1. When $n = 1, 2^{2 \cdot 1 + 1} + 1 = 2^3 + 1 = 8 + 1 = 9$ and 3 is a factor.
2. Assume 3 is a factor of $2^{2k+1} + 1$.

   Then,

$$\begin{aligned} 2^{2(k+1)+1} + 1 &= 2^{2k+2+1} + 1 \\ &= 2^{(2k+1)+2} + 1 \\ &= 2^{2k+1} \cdot 2^2 + 1 \\ &= 4 \cdot 2^{2k+1} + 1 \\ &= 4(2^{2k+1} + 1) - 3 \end{aligned}$$

   Since 3 is a factor of each term, 3 is a factor of the sum.

Thus, 3 is a factor of $2^{2n+1} + 1$ for all positive integers n.

33. A factor of $2^{4n-2} + 1$ is 5.

1. When $n = 1$,

   $2^{4(1)-2} + 1 = 5$ and 5 is a factor.
2. Assume that 5 is a factor of $2^{4k-2} + 1$.

   Then,

$$\begin{aligned} 2^{4(k+1)-2} + 1 &= 2^{4k+4-2} + 1 \\ &= 2^{4k-2} \cdot 2^4 + 1 \\ &= 2^{4k-2} \cdot 16 + 1 \\ &= (2^{4k-2} + 1) + 15 \cdot 2^{4k-2}. \end{aligned}$$

   Since 5 is a factor of $2^{4k-2} + 1$, our assumption, and 5 is a factor of $15 \cdot 2^{4k-2}$, we conclude that 5 is a factor of the entire expression.

Thus, 5 is a factor of $2^{4n-2} + 1$ for every positive integer n.

34. 1. When $n = 1$, $(2^{2(1)-1} + 3^{2(1)-1}) = 2 + 3 = 5$ and 5 is a factor.

2. Assume that 5 is a factor of $(2^{2k-1} + 3^{2k-1})$.

   Then,
$$\begin{aligned} 2^{2(k+1)-1} + 3^{2(k+1)-1} &= 2^{2k+2-1} + 3^{2k+2-1} \\ &= 2^{2k-1}2^2 + 3^{2k-1}3^2 \\ &= 4 \cdot 2^{2k-1} + 9 \cdot 3^{2k-1} \\ &= (2^{2k-1} + 3^{2k-1}) + (2^{2k-1} + 3^{2k-1}) \\ &\quad + (2^{2k-1} + 3^{2k-1}) + (2^{2k-1} + 3^{2k-1}) + 5 \cdot 3^{2k-1}. \end{aligned}$$

   Since 5 is a factor of each set in parentheses and 5 is a factor of $5 \cdot 3^{2k-1}$, then 5 is a factor of the whole sum.

Thus, 5 is a factor of $(2^{2n-1} + 3^{2n-1})$ for every positive integer n.

35. $S_n = 1 + 5 + 9 + 13 + \cdots + (4n - 3)$

$S_1 = 1 = 1 \cdot 1$

$S_2 = 1 + 5 = 6 = 2 \cdot 3$

$S_3 = 1 + 5 + 9 = 15 = 3 \cdot 5$

$S_4 = 1 + 5 + 9 + 13 = 28 = 4 \cdot 7$

From this sequence, it appears that $S_n = n(2n - 1)$. This can be verified by mathematical induction. The formula has already been verified for $n = 1$. Assume that the formula is valid for $n = k$. Then,

$$\begin{aligned} S_{k+1} &= [1 + 5 + 9 + 13 + \cdots + (4k - 3)] + [4(k + 1) - 3] \\ &= k(2k - 1) + (4k + 1) \\ &= 2k^2 + 3k + 1 \\ &= (k + 1)(2k + 1) \\ &= (k + 1)[2(k + 1) - 1]. \end{aligned}$$

Thus, the formula is valid.

36. $S_n = 25 + 22 + 19 + 16 + \cdots + (-3n + 28)$

$S_1 = 25 = \frac{1}{2}(50)$

$S_2 = 25 + 22 = 47 = \frac{2}{2}(47)$

$S_3 = 25 + 22 + 19 = 66 = \frac{3}{2}(44)$

$S_4 = 25 + 22 + 19 + 16 = 82 = \frac{4}{2}(41)$

From the sequence, it appears that

$$S_n = \frac{n}{2}(-3n + 53).$$

This can be verified by mathematical induction. The formula has already been verified for $n = 1$. Assume that the formula is valid for $n = k$. Then,

$$\begin{aligned} S_{k+1} &= [25 + 22 + 19 + 16 + \cdots + (-3k + 28)] + [-3(k + 1) + 28] \\ &= \frac{k}{2}(-3k + 53) + (-3k + 25) \\ &= \frac{1}{2}(-3k^2 + 47k + 50) \\ &= -\frac{1}{2}(3k^2 - 47k - 50) \\ &= -\frac{1}{2}(k + 1)(3k - 50) \\ &= \frac{k + 1}{2}[-3(k + 1) + 53]. \end{aligned}$$

Thus, the formula is valid.

37. $S_n = 1 + \frac{9}{10} + \frac{81}{100} + \frac{729}{1000} + \cdots + \left(\frac{9}{10}\right)^{n-1}$

Since this series is geometric, we have

$$S_n = \sum_{i=1}^{n}\left(\frac{9}{10}\right)^{i-1} = \frac{1 - \left(\frac{9}{10}\right)^n}{1 - \frac{9}{10}} = 10\left[1 - \left(\frac{9}{10}\right)^n\right]$$

$$= 10 - 10\left(\frac{9}{10}\right)^n.$$

38. $S_n = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \cdots + 3\left(-\frac{3}{2}\right)^{n-1}$

Since the series is geometric, we have

$$S_n = \sum_{i=1}^{n} 3\left(-\frac{3}{2}\right)^{i-1} = 3\left[\frac{1 - \left(-\frac{3}{2}\right)^n}{1 - \left(-\frac{3}{2}\right)}\right] = \frac{6}{5}\left[1 - \left(-\frac{3}{2}\right)^n\right].$$

39. $S_n = \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \frac{1}{40} + \cdots + \frac{1}{2n(n+1)}$

$$S_1 = \frac{1}{4} = \frac{1}{2(2)}$$

$$S_2 = \frac{1}{4} + \frac{1}{12} = \frac{4}{12} = \frac{2}{6} = \frac{2}{2(3)}$$

$$S_3 = \frac{1}{4} + \frac{1}{12} + \frac{1}{24} = \frac{9}{24} = \frac{3}{8} = \frac{3}{2(4)}$$

$$S_4 = \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \frac{1}{40} = \frac{16}{40} = \frac{4}{10} = \frac{4}{2(5)}$$

From this sequence, it appears that

$$S_n = \frac{n}{2(n+1)}.$$

This can be verified by mathematical induction. The formula has already been verified for $n = 1$. Assume that the formula is valid for $n = k$. Then,

$$S_{k+1} = \left[\frac{1}{4} + \frac{1}{12} + \frac{1}{40} + \cdots + \frac{1}{2k(k+1)}\right] + \frac{1}{2(k+1)(k+2)}$$

$$= \frac{k}{2(k+1)} + \frac{1}{2(k+1)(k+2)}$$

$$= \frac{k(k+2) + 1}{2(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{2(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{2(k+1)(k+2)}$$

$$= \frac{k+1}{2(k+2)}.$$

Thus, the formula is valid.

40. $S_n = \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots + \frac{1}{(n+1)(n+2)}$

$$S_1 = \frac{1}{6} = \frac{1}{2 \cdot 3}$$

$$S_2 = \frac{1}{6} + \frac{1}{12} = \frac{1}{4} = \frac{2}{2 \cdot 4}$$

$$S_3 = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{3}{10} = \frac{3}{2 \cdot 5}$$

$$S_4 = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{1}{3} = \frac{4}{2 \cdot 6}$$

From this sequence, it appears that

$$S_n = \frac{n}{2(n+2)}.$$

This can be verified by mathematical induction. The formula has already been verified for $n = 1$. Assume that the formula is valid for $n = k$. Then,

$$\begin{aligned} S_{k+1} &= \left[\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots + \frac{1}{(k+1)(k+2)}\right] + \frac{1}{(k+2)(k+3)} \\ &= \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)} \\ &= \frac{k(k+3)+2}{2(k+2)(k+3)} \\ &= \frac{k^2+3k+2}{2(k+2)(k+3)} \\ &= \frac{(k+1)(k+2)}{2(k+2)(k+3)} \\ &= \frac{k+1}{2[(k+1)+2]}. \end{aligned}$$

Thus, the formula is valid.

41. $\sum_{n=1}^{15} n = \frac{15(15+1)}{2} = 120$

42. $\sum_{n=1}^{30} n = \frac{30(30+1)}{2} = 465$

43. $\sum_{n=1}^{6} n^2 = \frac{6(6+1)[2(6)+1]}{6} = 91$

44. $\sum_{n=1}^{10} n^3 = \frac{10^2(10+1)^2}{4} = 3025$

45. $\sum_{n=1}^{5} n^4 = \frac{5(5+1)[2(5)+1][3(5)^2+3(5)-1]}{30} = 979$

46. $\sum_{n=1}^{8} n^5 = \frac{8^2(8+1)^2(2(8)^2+2(8)-1)}{12} = 61{,}776$

47. $$\begin{aligned} \sum_{n=1}^{6} (n^2 - n) &= \sum_{n=1}^{6} n^2 - \sum_{n=1}^{6} n \\ &= \frac{6(6+1)[2(6)+1]}{6} - \frac{6(6+1)}{2} \\ &= 91 - 21 = 70 \end{aligned}$$

48. $$\begin{aligned} \sum_{n=1}^{20} (n^3 - n) &= \sum_{n=1}^{20} n^3 - \sum_{n=1}^{20} n \\ &= \frac{(20)^2(20+1)^2}{4} - \frac{20(20+1)}{2} \\ &= \frac{(20)^2(21)^2 - 2(20)(21)}{4} = 43{,}890 \end{aligned}$$

49. $\sum_{i=1}^{6} (6i - 8i^3) = 6\sum_{i=1}^{6} i - 8\sum_{i=1}^{6} i^3 = 6\left[\frac{6(6+1)}{2}\right] - 8\left[\frac{(6)^2(6+1)^2}{4}\right] = 6(21) - 8(441) = -3402$

50. $$\sum_{j=1}^{10}\left(3 - \frac{1}{2}j + \frac{1}{2}j^2\right) = \sum_{j=1}^{10} 3 - \frac{1}{2}\sum_{j=1}^{10} j + \frac{1}{2}\sum_{j=1}^{10} j^2$$

$$= 3(10) - \frac{1}{2}\cdot\frac{10(10+1)}{2} + \frac{1}{2}\cdot\frac{10(10+1)(2\cdot 10+1)}{6}$$

$$= \frac{3(10)(12) - 3(10)(11) + 10(11)(21)}{12} = 195$$

51. $a_1 = 0, a_n = a_{n-1} + 3$

$a_1 = a_1 = 0$

$a_2 = a_1 + 3 = 0 + 3 = 3$

$a_3 = a_2 + 3 = 3 + 3 = 6$

$a_4 = a_3 + 3 = 6 + 3 = 9$

$a_5 = a_4 + 3 = 9 + 3 = 12$

$a_6 = a_5 + 3 = 12 + 3 = 15$

a_n: 0 3 6 9 12 15

First differences: 3 3 3 3 3

Second differences: 0 0 0 0

Since the first differences are equal, the sequence has a linear model.

52. $a_1 = 2, a_n = a_{n-1} + 2$

$a_1 = a_1 = 2$

$a_2 = a_1 + 2 = 2 + 2 = 4$

$a_3 = a_2 + 2 = 4 + 2 = 6$

$a_4 = a_3 + 2 = 6 + 2 = 8$

$a_5 = a_4 + 2 = 8 + 2 = 10$

$a_6 = a_5 + 2 = 10 + 2 = 12$

a_n: 2 4 6 8 10 12

First differences: 2 2 2 2 2

Second differences: 0 0 0 0

Since the first differences are equal, the sequence has a linear model.

53. $a_1 = 3, a_n = a_{n-1} - n$

$a_1 = a_1 = 3$

$a_2 = a_1 - 2 = 3 - 2 = 1$

$a_3 = a_2 - 3 = 1 - 3 = -2$

$a_4 = a_3 - 4 = -2 - 4 = -6$

$a_5 = a_4 - 5 = -6 - 5 = -11$

$a_6 = a_5 - 6 = -11 - 6 = -17$

a_n: 3 1 −2 −6 −11 −17

First differences: −2 −3 −4 −5 −6

Second differences: −1 −1 −1 −1

Since the second differences are all the same, the sequence has a quadratic model.

54. $a_2 = -3, a_n = -2a_{n-1}$

$a_2 = -3 \Rightarrow -3 = -2a_1$

$a_1 = \frac{3}{2}$

$a_2 = -3$

$a_3 = -2a_2 = -2(-3) = 6$

$a_4 = -2a_3 = -2(6) = -12$

$a_5 = -2a_4 = -2(-12) = 24$

$a_6 = -2a_5 = -2(24) = -48$

$a_7 = -2a_6 = -2(-48) = 96$

a_n: $\frac{3}{2}$ −3 6 −12 24 −48 96

First differences: $-\frac{9}{2}$ 9 −18 36 −72 144

Second differences: $\frac{27}{2}$ −27 54 −108 216

Since neither the first differences nor the second differences are equal, the sequence does not have a linear or quadratic model.

55. $a_0 = 2, a_n = (a_{n-1})^2$

$a_0 = 2$

$a_1 = a_0{}^2 = 2^2 = 4$

$a_2 = a_1{}^2 = 4^2 = 16$

$a_3 = a_2{}^2 = 16^2 = 256$

$a_4 = a_3{}^2 = 256^2 = 65{,}536$

$a_5 = a_4{}^2 = 65{,}536^2 = 4{,}294{,}967{,}296$

a_n: 2, 4, 16, 256, 65,536, 4,294,967,296

First differences: 2, 12, 240, 65,280, 4,294,901,760

Second differences: 10, 228, 65,040, 4,294,836,480

Since neither the first differences nor the second differences are equal, the sequence does not have a linear or quadratic model.

56. $a_0 = 0, a_n = a_{n-1} + n$

$a_0 = 0$

$a_1 = a_0 + 1 = 0 + 1 = 1$

$a_2 = a_1 + 2 = 1 + 2 = 3$

$a_3 = a_2 + 3 = 3 + 3 = 6$

$a_4 = a_3 + 4 = 6 + 4 = 10$

$a_5 = a_4 + 5 = 10 + 5 = 15$

a_n: 0, 1, 3, 6, 10, 15

First differences: 1, 2, 3, 4, 5

Second differences: 1, 1, 1, 1

Since the second differences are equal, the sequence has a quadratic model.

57. $a_0 = 3, a_1 = 3, a_4 = 15$

Let $a_n = an^2 + bn + c$.

Thus: $a_0 = a(0)^2 + b(0) + c = 3 \implies c = 3$

$a_1 = a(1)^2 + b(1) + c = 3 \implies a + b + c = 3$

$a + b = 0$

$a_4 = a(4)^2 + b(4) + c = 15 \implies 16a + 4b + c = 15$

$16a + 4b = 12$

$4a + b = 3$

By elimination:
$$\begin{array}{r} -a - b = 0 \\ \underline{4a + b = 3} \\ 3a \quad = 3 \end{array}$$

$a = 1 \implies b = -1$

Thus, $a_n = n^2 - n + 3$.

58. $a_0 = 7, a_1 = 6, a_3 = 10$

Let $a_n = an^2 + bn + c$. Then:

$a_0 = a(0)^2 + b(0) + c = 7 \implies c = 7$

$a_1 = a(1)^2 + b(1) + c = 6 \implies a + b + c = 6$

$a + b = -1$

$a_3 = a(3)^2 + b(3) + c = 10 \implies 9a + 3b + c = 10$

$9a + 3b = 3$

$3a + b = 1$

By elimination:
$$\begin{array}{r} -a - b = 1 \\ \underline{3a + b = 1} \\ 2a = 2 \end{array}$$

$a = 1 \implies b = -2$

Thus, $a_n = n^2 - 2n + 7$.

59. $a_0 = -3, a_2 = 1, a_4 = 9$

Let $a_n = an^2 + bn + c$.

Then: $a_0 = a(0)^2 + b(0) + c = -3 \implies c = -3$

$a_2 = a(2)^2 + b(2) + c = 1 \implies 4a + 2b + c = 1$

$4a + 2b = 4$

$2a + b = 2$

$a_4 = a(4)^2 + b(4) + c = 9 \implies 16a + 4b + c = 9$

$16a + 4b = 12$

$4a + b = 3$

By elimination:
$$\begin{array}{r} -2a - b = -2 \\ \underline{4a + b = 3} \\ 2a \quad = 1 \end{array}$$

$a = \frac{1}{2} \implies b = 1$

Thus, $a_n = \frac{1}{2}n^2 + n - 3$.

60. $a_0 = 3, a_2 = 0, a_6 = 36$

Let $a_n = an^2 + bn + c$. Then:

$$\begin{aligned}
a_0 &= a(0)^2 + b(0) + c = 3 \implies & c &= 3\\
a_2 &= a(2)^2 + b(2) + c = 0 \implies & 4a + 2b + c &= 0\\
& & 4a + 2b &= -3\\
a_6 &= a(6)^2 + b(6) + c = 36 \implies & 36a + 6b + c &= 36\\
& & 36a + 6b &= 33\\
& & 12a + 2b &= 11
\end{aligned}$$

By elimination:
$$\begin{aligned}
-4a - 2b &= 3\\
12a + 2b &= 11\\
\hline
8a &= 14\\
a &= \tfrac{7}{4} \implies b = -5
\end{aligned}$$

Thus, $a_n = \frac{7}{4}n^2 - 5n + 3$.

61. (a)

	120.3		122.5		124.9		127.1		129.4		130.3
First differences:		2.2		2.4		2.2		2.3		0.9	

(b) The first differences are not equal, but are fairly close to each other, so a linear model can be used. If we let $m = 2.2$, then $b = 120.3 - 2.2(8) = 102.7$

$a_n \approx 2.2n + 102.7$

(c) $a_n \approx 2.08n + 103.9$ is obtained by using the regression feature of a graphing utility.

(d) For 2008, let $n = 18$.

$a_n \approx 2.2(18) + 102.7 = 142.3$

$a_n = 2.08(18) + 103.9 = 141.34$

These are very similar.

62. Answers will vary. See page 626.

63. True. P_7 may be false.

64. False. P_1 must be proven to be true.

65. True. If the second differences are all zero, then the first differences are all the same, so the sequence is arithmetic.

66. False. It has $n - 2$ second differences.

67. $(2x^2 - 1)^2 = (2x^2 - 1)(2x^2 - 1) = 4x^4 - 4x^2 + 1$

68. $(2x - y)^2 = 4x^2 - 4xy + y^2$

69. $(5 - 4x)^3 = -64x^3 + 240x^2 - 300x + 125$

70. $(2x - 4y)^3 = 8x^3 - 48x^2y + 96xy^2 - 64y^3$

71. $f(x) = \dfrac{x}{x + 3}$

(a) Domain: All real numbers x except $x = -3$

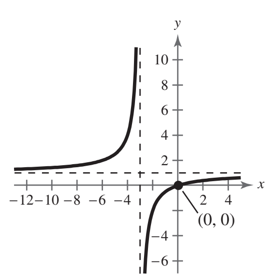
(b) Intercept: $(0, 0)$

(c) Vertical asymptote: $x = -3$

Horizontal asymptote: $y = 1$

(d)

x	-5	-4	-2	-1	1
$f(x)$	$\frac{5}{2}$	4	-2	$-\frac{1}{2}$	$\frac{1}{4}$



72. $g(x) = \dfrac{x^2}{x^2 - 4}$

(a) Domain: All real numbers x except $x = \pm 2$

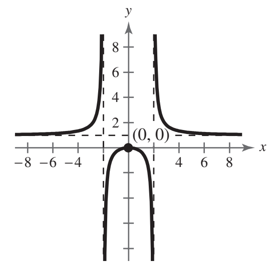
(b) Intercept: $(0, 0)$

(c) Vertical asymptotes: $x = -2, x = 2$

Horizontal asymptote: $y = 1$

(d)

x	-4	-3	-1.5	0	1.5	3	4
$g(x)$	$\frac{4}{3}$	$\frac{9}{5}$	$-\frac{9}{7}$	0	$-\frac{9}{7}$	$\frac{9}{5}$	$\frac{4}{3}$



73. $h(t) = \dfrac{t - 7}{t}$

(a) Domain: All real numbers t except $t = 0$

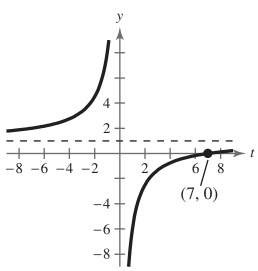
(b) Intercept: $(7, 0)$

(c) Vertical asymptote: $t = 0$

Horizontal asymptote: $y = 1$

(d)

t	-2	-1	1	2	3
$h(t)$	$\frac{9}{2}$	8	-6	$-\frac{5}{2}$	$-\frac{4}{3}$



74. $f(x) = \dfrac{5 + x}{1 - x}$

(a) Domain: All real numbers x except $x = 1$

(b) x-intercept: $(-5, 0)$

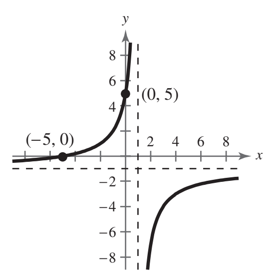
y-intercept: $(0, 5)$

(c) Vertical asymptote: $x = 1$

Horizontal asymptote: $y = -1$

(d)

x	-8	-5	-2	0	2	3	5	7
$f(x)$	$-\frac{1}{3}$	0	1	5	-7	-4	$-\frac{5}{2}$	-2



Section 9.5 The Binomial Theorem

- You should be able to use the formula

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdot \cdot \cdot + {}_nC_r x^{n-r}y^r + \cdot \cdot \cdot + y^n$$

where ${}_nC_r = \dfrac{n!}{(n-r)!r!}$, to expand $(x + y)^n$. Also, ${}_nC_r = \dbinom{n}{r}$.

- You should be able to use Pascal's Triangle in binomial expansion.

Vocabulary Check

1. binomial coefficients

2. Binomial Theorem/Pascal's Triangle

3. $\dbinom{n}{r}$ or ${}_nC_r$

4. expanding a binomial

1. $_5C_3 = \dfrac{5!}{3!2!} = \dfrac{5 \cdot 4}{2 \cdot 1} = 10$

2. $_8C_6 = \dfrac{8!}{6! \cdot 2!} = \dfrac{8 \cdot 7}{2 \cdot 1} = 28$

3. $_{12}C_0 = \dfrac{12!}{0!12!} = 1$

4. $_{20}C_{20} = \dfrac{20!}{20! \cdot 0!} = 1$

5. $_{20}C_{15} = \dfrac{20!}{15!5!} = \dfrac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 15{,}504$

6. $_{12}C_5 = \dfrac{12!}{5! \cdot 7!} = \dfrac{(12 \cdot 11 \cdot 10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{5!\cancel{7!}} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 792$

7. $\dbinom{10}{4} = \dfrac{10!}{6!4!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!(24)} = 210$

8. $\dbinom{10}{6} = \dfrac{10!}{6! \cdot 4!} = \dfrac{(10 \cdot 9 \cdot 8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 4!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$

9. $\dbinom{100}{98} = \dfrac{100!}{2!98!} = \dfrac{100 \cdot 99}{2 \cdot 1} = 4950$

10. $\dbinom{100}{2} = \dfrac{100!}{98! \cdot 2!} = \dfrac{(100 \cdot 99) \cdot \cancel{98!}}{\cancel{98!} \cdot 2!} = \dfrac{100 \cdot 99}{2 \cdot 1}$

$= 4950$

11.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}$$

$\dbinom{8}{5} = 56$, the 6th entry in the 8th row.

12.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}$$

$\dbinom{8}{7} = 8$, the 8th entry in the 8th row.

13.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}$$

$_7C_4 = 35$, the 5th entry in the 7th row.

14.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}$$

$_6C_3 = 20$, the 4th entry in the 6th row.

15. $(x + 1)^4 = {_4C_0}x^4 + {_4C_1}x^3(1) + {_4C_2}x^2(1)^2 + {_4C_3}x(1)^3 + {_4C_4}(1)^4$

$= x^4 + 4x^3 + 6x^2 + 4x + 1$

16. $(x + 1)^6 = {_6C_0}x^6 + {_6C_1}x^5(1) + {_6C_2}x^4(1)^2 + {_6C_3}x^3(1)^3 + {_6C_4}x^2(1)^4 + {_6C_5}x(1)^5 + {_6C_6}(1)^6$

$= x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

17. $(a + 6)^4 = {_4C_0}a^4 + {_4C_1}a^3(6) + {_4C_2}a^2(6)^2 + {_4C_3}a(6)^3 + {_4C_4}(6)^4$

$= 1a^4 + 4a^3(6) + 6a^2(6)^2 + 4a(6)^3 + 1(6)^4$

$= a^4 + 24a^3 + 216a^2 + 864a + 1296$

18. $(a+5)^5 = {}_5C_0a^5 + {}_5C_1a^4(5) + {}_5C_2a^3(5)^2 + {}_5C_3a^2(5)^3 + {}_5C_4a(5)^4 + {}_5C_5(5)^5$

$= a^5 + 25a^4 + 250a^3 + 1250a^2 + 3125a + 3125$

19. $(y-4)^3 = {}_3C_0y^3 - {}_3C_1y^2(4) + {}_3C_2y(4)^2 - {}_3C_3(4)^3$

$= 1y^3 - 3y^2(4) + 3y(4)^2 - 1(4)^3$

$= y^3 - 12y^2 + 48y - 64$

20. $(y-2)^5 = {}_5C_0y^5 - {}_5C_1y^4(2) + {}_5C_2y^3(2)^2 - {}_5C_3y^2(2)^3 + {}_5C_4y(2)^4 - {}_5C_5(2)^5$

$= y^5 - 10y^4 + 40y^3 - 80y^2 + 80y - 32$

21. $(x+y)^5 = {}_5C_0x^5 + {}_5C_1x^4y + {}_5C_2x^3y^2 + {}_5C_3x^2y^3 + {}_5C_4xy^4 + {}_5C_5y^5$

$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

22. $(c+d)^3 = {}_3C_0c^3 + {}_3C_1c^2d + {}_3C_2cd^2 + {}_3C_3d^3$

$= c^3 + 3c^2d + 3cd^2 + d^3$

23. $(r+3s)^6 = {}_6C_0r^6 + {}_6C_1r^5(3s) + {}_6C_2r^4(3s)^2 + {}_6C_3r^3(3s)^3 + {}_6C_4r^2(3s)^4 + {}_6C_5r(3s)^5 + {}_6C_6(3s)^6$

$= 1r^6 + 6r^5(3s) + 15r^4(3s)^2 + 20r^3(3s)^3 + 15r^2(3s)^4 + 6r(3s)^5 + 1(3s)^6$

$= r^6 + 18r^5s + 135r^4s^2 + 540r^3s^3 + 1215r^2s^4 + 1458rs^5 + 729s^6$

24. $(x+2y)^4 = {}_4C_0x^4 + {}_4C_1x^3(2y) + {}_4C_2x^2(2y)^2 + {}_4C_3x(2y)^3 + {}_4C_4(2y)^4$

$= x^4 + 4x^3(2y) + 6x^2(4y^2) + 4x(8y^3) + 16y^4$

$= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$

25. $(3a-4b)^5 = {}_5C_0(3a)^5 - {}_5C_1(3a)^4(4b) + {}_5C_2(3a)^3(4b)^2 - {}_5C_3(3a)^2(4b)^3 + {}_5C_4(3a)(4b)^4 - {}_5C_5(4b)^5$

$= (1)(243a^5) - 5(81a^4)(4b) + 10(27a^3)(16b^2) - 10(9a^2)(64b^3) + 5(3a)(256b^4) - (1)(1024b^5)$

$= 243a^5 - 1620a^4b + 4320a^3b^2 - 5760a^2b^3 + 3840ab^4 - 1024b^5$

26. $(2x-5y)^5 = {}_5C_0(2x)^5 + {}_5C_1(2x)^4(-5y) + {}_5C_2(2x)^3(-5y)^2 + {}_5C_3(2x)^2(-5y)^3 + {}_5C_4(2x)(-5y)^4 + {}_5C_5(-5y)^5$

$= (2x)^5 + 5(2x)^4(-5y) + 10(2x)^3(-5y)^2 + 10(2x)^2(-5y)^3 + 5(2x)(-5y)^4 + (-5y)^5$

$= 32x^5 - 400x^4y + 2000x^3y^2 - 5000x^2y^3 + 6250xy^4 - 3125y^5$

27. $(2x+y)^3 = {}_3C_0(2x)^3 + {}_3C_1(2x)^2(y) + {}_3C_2(2x)(y^2) + {}_3C_3(y^3)$

$= (1)(8x^3) + (3)(4x^2)(y) + (3)(2x)(y^2) + (1)(y^3)$

$= 8x^3 + 12x^2y + 6xy^2 + y^3$

28. $(7a+b)^3 = {}_3C_0(7a)^3 + {}_3C_1(7a)^2(b) + {}_3C_2(7a)(b)^2 + {}_3C_3(b)^3$

$= (7a)^3 + 3(7a)^2(b) + 3(7a)(b)^2 + (b)^3$

$= 343a^3 + 147a^2b + 21ab^2 + b^3$

29. $(x^2 + y^2)^4 = {}_4C_0(x^2)^4 + {}_4C_1(x^2)^3(y^2) + {}_4C_2(x^2)^2(y^2)^2 + {}_4C_3(x^2)(y^2)^3 + {}_4C_4(y^2)^4$

$= (1)(x^8) + (4)(x^6y^2) + (6)(x^4y^4) + (4)(x^2y^6) + (1)(y^8)$

$= x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8$

30. $(x^2 + y^2)^6 = {}_6C_0(x^2)^6 + {}_6C_1(x^2)^5(y^2) + {}_6C_2(x^2)^4(y^2)^2 + {}_6C_3(x^2)^3(y^2)^3 + {}_6C_4(x^2)^2(y^2)^4 + {}_6C_5(x^2)(y^2)^5 + {}_6C_6(y^2)^6$

$= x^{12} + 6x^{10}y^2 + 15x^8y^4 + 20x^6y^6 + 15x^4y^8 + 6x^2y^{10} + y^{12}$

31. $\left(\frac{1}{x} + y\right)^5 = {}_5C_0\left(\frac{1}{x}\right)^5 + {}_5C_1\left(\frac{1}{x}\right)^4 y + {}_5C_2\left(\frac{1}{x}\right)^3 y^2 + {}_5C_3\left(\frac{1}{x}\right)^2 y^3 + {}_5C_4\left(\frac{1}{x}\right)y^4 + {}_5C_5y^5$

$= \frac{1}{x^5} + \frac{5y}{x^4} + \frac{10y^2}{x^3} + \frac{10y^3}{x^2} + \frac{5y^4}{x} + y^5$

32. $\left(\frac{1}{x} + 2y\right)^6 = {}_6C_0\left(\frac{1}{x}\right)^6 + {}_6C_1\left(\frac{1}{x}\right)^5(2y) + {}_6C_2\left(\frac{1}{x}\right)^4(2y)^2 + {}_6C_3\left(\frac{1}{x}\right)^3(2y)^3 + {}_6C_4\left(\frac{1}{x}\right)^2(2y)^4 + {}_6C_5\left(\frac{1}{x}\right)(2y)^5 + {}_6C_6(2y)^6$

$= 1\left(\frac{1}{x}\right)^6 + 6(2)\left(\frac{1}{x}\right)^5 y + 15(4)\left(\frac{1}{x}\right)^4 y^2 + 20(8)\left(\frac{1}{x}\right)^3 y^3 + 15(16)\left(\frac{1}{x}\right)^2 y^4 + 6(32)\left(\frac{1}{x}\right)y^5 + 1(64)y^6$

$= \frac{1}{x^6} + \frac{12y}{x^5} + \frac{60y^2}{x^4} + \frac{160y^3}{x^3} + \frac{240y^4}{x^2} + \frac{192y^5}{x} + 64y^6$

33. $2(x - 3)^4 + 5(x - 3)^2 = 2[x^4 - 4(x^3)(3) + 6(x^2)(3^2) - 4(x)(3^3) + 3^4] + 5[x^2 - 2(x)(3) + 3^2]$

$= 2(x^4 - 12x^3 + 54x^2 - 108x + 81) + 5(x^2 - 6x + 9)$

$= 2x^4 - 24x^3 + 113x^2 - 246x + 207$

34. $3(x + 1)^5 - 4(x + 1)^3 = 3[{}_5C_0x^5 + {}_5C_1x^4(1) + {}_5C_2x^3(1)^2 + {}_5C_3x^2(1)^3 + {}_5C_4x(1)^4 + {}_5C_5(1)^5]$

$- 4[{}_3C_0x^3 + {}_3C_1x^2(1) + {}_3C_2x(1)^2 + {}_3C_3(1)^3]$

$= 3[(1)x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1] - 4[(1)x^3 + 3x^2 + 3x + 1]$

$= 3x^5 + 15x^4 + 26x^3 + 18x^2 + 3x - 1$

35. 5th Row of Pascal's Triangle: 1 5 10 10 5 1

$(2t - s)^5 = 1(2t)^5 - 5(2t)^4(s) + 10(2t)^3(s)^2 - 10(2t)^2(s)^3 + 5(2t)(s)^4 - 1(s)^5$

$= 32t^5 - 80t^4s + 80t^3s^2 - 40t^2s^3 + 10ts^4 - s^5$

36. 4th Row of Pascal's Triangle: 1 4 6 4 1

$(3 - 2z)^4 = 3^4 - 4(3)^3(2z) + 6(3)^2(2z)^2 - 4(3)(2z)^3 + (2z)^4$

$= 81 - 216z + 216z^2 - 96z^3 + 16z^4$

37. 5th Row of Pascal's Triangle: 1 5 10 10 5 1

$(x + 2y)^5 = 1x^5 + 5x^4(2y) + 10x^3(2y)^2 + 10x^2(2y)^3 + 5x(2y)^4 + 1(2y)^5$

$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

38. 6th Row of Pascal's Triangle: 1 6 15 20 15 6 1

$(2v + 3)^6 = (2v)^6 + 6(2v)^5(3) + 15(2v)^4(3)^2 + 20(2v)^3(3)^3 + 15(2v)^2(3)^4 + 6(2v)(3)^5 + (3)^6$

$= 64v^6 + 576v^5 + 2160v^4 + 4320v^3 + 4860v^2 + 2916v + 729$

39. The 4^{th} term in the expansion of $(x + y)^{10}$ is

$_{10}C_3 x^{10-3}y^3 = 120x^7y^3.$

40. The 7^{th} term in the expansion of $(x - y)^6$ is

$_6C_6 x^{6-6}(-y)^6 = 1 \cdot x^0y^6 = y^6.$

41. The 3^{rd} term in the expansion of $(x - 6y)^5$ is

$_5C_2 x^{5-2}(-6y)^2 = 10x^3(36y^2) = 360x^3y^2.$

42. The 4^{th} term in the expansion of $(x - 10z)^7$ is

$_7C_3 x^{7-3}(-10z)^3 = 35 \cdot x^4(-1000z^3) = -35{,}000x^4z^3.$

43. The 8^{th} term in the expansion of $(4x + 3y)^9$ is

$$_9C_7(4x)^{9-7}(3y)^7 = 36(16x^2)(2187y^7)$$
$$= 1{,}259{,}712x^2y^7.$$

44. The 5^{th} term in the expansion of $(5a + 6b)^5$ is

$_5C_4(5a)^{5-4}(6b)^4 = 5 \cdot (5a)(1296b^4) = 32{,}400ab^4.$

45. The 9^{th} term in the expansion of $(10x - 3y)^{12}$ is

$$_{12}C_8(10x)^{12-8}(-3y)^8 = 495(10{,}000x^4)(6561y^8)$$
$$= 32{,}476{,}950{,}000x^4y^8.$$

46. The 7^{th} term in the expansion of $(7x + 2y)^{15}$ is

$$_{15}C_6(7x)^{15-6}(2y)^6 = 5005 \cdot (40{,}353{,}607x^9)(64y^6)$$
$$\approx 1.293 \times 10^{13}x^9y^6.$$

47. The term involving x^5 in the expansion of $(x + 3)^{12}$ is

$$_{12}C_7x^5(3)^7 = \frac{12!}{7!5!} \cdot 3^7x^5 = 1{,}732{,}104x^5.$$

The coefficient is 1,732,104.

48. The term involving x^8 in the expansion of $(x^2 + 3)^{12}$ is

$$_{12}C_8(x^2)^4(3)^8 = \frac{12!}{(12-8)!8!} \cdot 3^8x^8 = 3{,}247{,}695x^8.$$

The coefficient is 3,247,695.

49. The term involving x^8y^2 in the expansion of $(x - 2y)^{10}$ is

$$_{10}C_2x^8(-2y)^2 = \frac{10!}{2!8!} \cdot 4x^8y^2 = 180x^8y^2.$$

The coefficient is 180.

50. The term involving x^2y^8 in the expansion of $(4x - y)^{10}$ is

$$_{10}C_8(4x)^2(-y)^8 = \frac{10!}{(10-8)!8!} \cdot 16x^2y^8 = 720x^2y^8.$$

The coefficient is 720.

51. The term involving x^4y^5 in the expansion of $(3x - 2y)^9$ is

$$_9C_5(3x)^4(-2y)^5 = \frac{9!}{5!4!}(81x^4)(-32y^5) = -326{,}592x^4y^5.$$

The coefficient is $-326{,}592$.

52. The term involving x^6y^2 in the expansion of $(2x - 3y)^8$ is

$$_8C_2(2x)^6(-3y)^2 = \frac{8!}{(8-2)!2!}(64x^6)(9y^2) = 16{,}128x^6y^2.$$

The coefficient is 16,128.

53. The term involving $x^8y^6 = (x^2)^4y^6$ in the expansion of

$(x^2 + y)^{10}$ is $_{10}C_6(x^2)^4y^6 = \dfrac{10!}{4!6!}(x^2)^4y^6 = 210x^8y^6.$

The coefficient is 210.

54. The term involving z^4t^8 in the expansion of $(z^2 - t)^{10}$ is

$$_{10}C_8(z^2)^2(-t)^8 = \frac{10!}{(10-8)!8!}z^4t^8 = 45z^4t^8.$$

The coefficient is 45.

55.
$$\left(\sqrt{x} + 3\right)^4 = \left(\sqrt{x}\right)^4 + 4\left(\sqrt{x}\right)^3(3) + 6\left(\sqrt{x}\right)^2(3)^2 + 4\left(\sqrt{x}\right)(3)^3 + (3)^4$$
$$= x^2 + 12x\sqrt{x} + 54x + 108\sqrt{x} + 81$$
$$= x^2 + 12x^{3/2} + 54x + 108x^{1/2} + 81$$

56.
$$\left(2\sqrt{t} - 1\right)^3 = \left(2\sqrt{t}\right)^3 + 3\left(2\sqrt{t}\right)^2(-1) + 3\left(2\sqrt{t}\right)(-1)^2 + (-1)^3$$
$$= 8t^{3/2} - 12t + 6t^{1/2} - 1$$

57.
$$(x^{2/3} - y^{1/3})^3 = (x^{2/3})^3 - 3(x^{2/3})^2(y^{1/3}) + 3(x^{2/3})(y^{1/3})^2 - (y^{1/3})^3$$
$$= x^2 - 3x^{4/3}y^{1/3} + 3x^{2/3}y^{2/3} - y$$

58. $(u^{3/5} + 2)^5 = (u^{3/5})^5 + 5(u^{3/5})^4(2) + 10(u^{3/5})^3(2)^2 + 10(u^{3/5})^2(2)^3 + 5(u^{3/5})(2)^4 + 2^5$

$= u^3 + 10u^{12/5} + 40u^{9/5} + 80u^{6/5} + 80u^{3/5} + 32$

59. $$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \frac{h(3x^2 + 3xh + h^2)}{h} = 3x^2 + 3xh + h^2, h \neq 0$$

60. $$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^4 - x^4}{h} = \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} = \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3, h \neq 0$$

61. $$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{\sqrt{x+h} + \sqrt{x}}, h \neq 0$$

62. $$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \frac{\dfrac{x - (x+h)}{x(x+h)}}{h} = \frac{\dfrac{-h}{x(x+h)}}{h} = -\frac{1}{x(x+h)}, h \neq 0$$

63. $(1 + i)^4 = {}_4C_0(1)^4 + {}_4C_1(1)^3 i + {}_4C_2(1)^2 i^2 + {}_4C_3(1)i^3 + {}_4C_4 i^4$

$= 1 + 4i - 6 - 4i + 1$

$= -4$

64. $(2 - i)^5 = {}_5C_0 2^5 - {}_5C_1 2^4 i + {}_5C_2 2^3 i^2 - {}_5C_3 2^2 i^3 + {}_5C_4 2i^4 - {}_5C_5 i^5$

$= 32 - 80i - 80 + 40i + 10 - i$

$= -38 - 41i$

65. $(2 - 3i)^6 = {}_6C_0 2^6 - {}_6C_1 2^5(3i) + {}_6C_2 2^4(3i)^2 - {}_6C_3 2^3(3i)^3 + {}_6C_4 2^2(3i)^4 - {}_6C_5 2(3i)^5 + {}_6C_6(3i)^6$

$= (1)(64) - (6)(32)(3i) + 15(16)(-9) - 20(8)(-27i) + 15(4)(81) - 6(2)(243i) + (1)(-729)$

$= 64 - 576i - 2160 + 4320i + 4860 - 2916i - 729$

$= 2035 + 828i$

66. $\left(5 + \sqrt{-9}\right)^3 = (5 + 3i)^3$

$= 5^3 + 3 \cdot 5^2(3i) + 3 \cdot 5(3i)^2 + (3i)^3$

$= 125 + 225i - 135 - 27i$

$= -10 + 198i$

67. $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 = \frac{1}{8}\left[(-1)^3 + 3(-1)^2\left(\sqrt{3}i\right) + 3(-1)\left(\sqrt{3}i\right)^2 + \left(\sqrt{3}i\right)^3\right]$

$= \frac{1}{8}\left[-1 + 3\sqrt{3}i + 9 - 3\sqrt{3}i\right]$

$= 1$

68. $\left(5 - \sqrt{3}i\right)^4 = 5^4 - 4 \cdot 5^3\left(\sqrt{3}i\right) + 6 \cdot 5^2\left(\sqrt{3}i\right)^2 - 4 \cdot 5\left(\sqrt{3}i\right)^3 + \left(\sqrt{3}i\right)^4$

$= 625 - 500\sqrt{3}i - 450 + 60\sqrt{3}i + 9$

$= 184 - 440\sqrt{3}i$

69. $(1.02)^8 = (1 + 0.02)^8$

$= 1 + 8(0.02) + 28(0.02)^2 + 56(0.02)^3 + 70(0.02)^4 + 56(0.02)^5 + 28(0.02)^6 + 8(0.02)^7 + (0.02)^8$

$= 1 + 0.16 + 0.0112 + 0.000448 + \cdots \approx 1.172$

70. $(2.005)^{10} = (2 + 0.005)^{10} = 2^{10} + 10(2)^9(0.005) + 45(2)^8(0.005)^2 + 120(2)^7(0.005)^3 + 210(2)^6(0.005)^4$

$+ 252(2)^5(0.005)^5 + 210(2)^4(0.005)^6 + 120(2)^3(0.005)^7 + 45(2)^2(0.005)^8$

$+ 10(2)(0.005)^9 + (0.005)^{10}$

$= 1024 + 25.6 + 0.288 + 0.00192 + 0.0000084 + \cdots$

≈ 1049.890

71. $(2.99)^{12} = (3 - 0.01)^{12}$

$= 3^{12} - 12(3)^{11}(0.01) + 66(3)^{10}(0.01)^2 - 220(3)^9(0.01)^3 + 495(3)^8(0.01)^4$

$- 792(3)^7(0.01)^5 + 924(3)^6(0.01)^6 - 792(3)^5(0.01)^7 + 495(3)^4(0.01)^8$

$- 220(3)^3(0.01)^9 + 66(3)^2(0.01)^{10} - 12(3)(0.01)^{11} + (0.01)^{12}$

$\approx 531{,}441 - 21{,}257.64 + 389.7234 - 4.3303 + 0.0325 - 0.0002 + \cdots \approx 510{,}568.785$

72. $(1.98)^9 = (2 - 0.02)^9 = 2^9 - 9(2)^8(0.02) + 36(2)^7(0.02)^2 - 84(2)^6(0.02)^3 + 126(2)^5(0.02)^4$

$- 126(2)^4(0.02)^5 + 84(2)^3(0.02)^6 - 36(2)^2(0.02)^7 + 9(2)(0.02)^8 - (0.02)^9$

$= 512 - 46.08 + 1.8432 - 0.043008 + 0.00064512 + 0.0000064512 + \cdots$

≈ 467.721

73. $f(x) = x^3 - 4x$

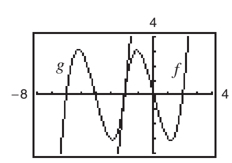
$g(x) = f(x + 4)$

$= (x + 4)^3 - 4(x + 4)$

$= x^3 + 3x^2(4) + 3x(4)^2 + (4)^3 - 4x - 16$

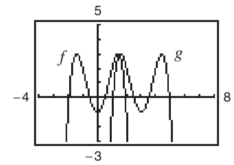
$= x^3 + 12x^2 + 48x + 64 - 4x - 16$

$= x^3 + 12x^2 + 44x + 48$



The graph of g is the same as the graph of f shifted four units to the left.

74. $f(x) = -x^4 + 4x^2 - 1,\ g(x) = f(x - 3)$



$$\begin{aligned} g(x) &= f(x - 3) \\ &= -(x - 3)^4 + 4(x - 3)^2 - 1 \\ &= -(x^4 + 4x^3(-3) + 6x^2(-3)^2 + 4x(-3)^3 + (-3)^4) + 4(x^2 - 6x + 9) - 1 \\ &= -x^4 + 12x^3 - 54x^2 + 108x - 81 + 4x^2 - 24x + 36 - 1 \\ &= -x^4 + 12x^3 - 50x^2 + 84x - 46 \end{aligned}$$

The graph of g is the same as the graph of f shifted three units to the right.

75. ${}_7C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^3 = \frac{7!}{3!4!}\left(\frac{1}{16}\right)\left(\frac{1}{8}\right) = 35\left(\frac{1}{16}\right)\left(\frac{1}{8}\right) \approx 0.273$

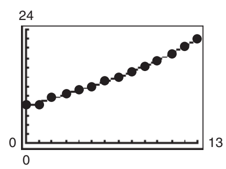
76. ${}_{10}C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7 = 120\left(\frac{1}{64}\right)\left(\frac{2187}{16{,}384}\right) \approx 0.2503$

77. ${}_8C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^4 = \frac{8!}{4!4!}\left(\frac{1}{81}\right)\left(\frac{16}{81}\right) = 70\left(\frac{1}{81}\right)\left(\frac{16}{81}\right) \approx 0.171$

78. ${}_8C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^4 = 70\left(\frac{1}{16}\right)\left(\frac{1}{16}\right) \approx 0.273$

79. (a) $f(t) \approx 0.0025t^3 - 0.015t^2 + 0.88t + 7.7$

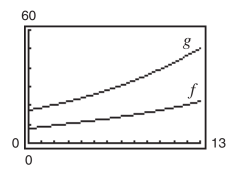
(b)



(c) $g(t) = f(t + 10) = 0.0025(t + 10)^3 - 0.015(t + 10)^2 + 0.88(t + 10) + 7.7$

$$\begin{aligned} &= 0.0025(t^3 + 30t^2 + 300t + 1000) \\ &\quad - 0.015(t^2 + 20t + 100) + 0.88(t + 10) + 7.7 \\ &= 0.0025t^3 + 0.06t^2 + 1.33t + 17.5 \end{aligned}$$

(d)



(e) For 2008 use $t = 18$ in $f(t)$ and $t = 8$ in $g(t)$.

$f(18) = 33.26$ gallons

$g(8) = 33.26$ gallons

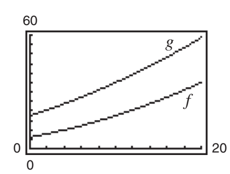
Both models yield the same answer.

(f) The trend is for the per capita consumption of bottled water to increase. This may be due to the increasing concern with contaminants in tap water.

80. $f(t) = 0.031t^2 + 0.82t + 6.1$

(a)
$$\begin{aligned} g(t) &= f(t + 10) \\ &= 0.031(t + 10)^2 + 0.82(t + 10) + 6.1 \\ &= 0.031(t^2 + 20t + 100) + 0.82(t + 10) + 6.1 \\ &= 0.031t^2 + 1.44t + 17.4 \end{aligned}$$

(b)



(c) $f(t)$: $f(17) = 2007$

$g(t)$: $g(7) = 2007$

81. True. The coefficients from the Binomial Theorem can be used to find the numbers in Pascal's Triangle.

82. False. Expanding binomials that represent differences is just as accurate as expanding binomials that represent sums, but for differences the coefficient signs are alternating.

83. False.

The coefficient of the x^{10}-term is ${}_{12}C_7(3)^7 = 1{,}732{,}104$.

The coefficient of the x^{14}-term is ${}_{12}C_5(3)^5 = 192{,}456$.

84. The first and last numbers in each row are 1. Every other number in each row is formed by adding the two numbers immediately above the number.

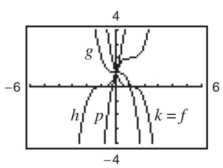
85.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

86. $(n + 1)$ terms

87. The signs of the terms in the expansion of $(x - y)^n$ alternate from positive to negative.

88. The functions $f(x) = (1 - x)^3$ and $k(x) = 1 - 3x + 3x^2 + x^3$ (choices (a) and (d)) have identical graphs, because $k(x)$ is the expansion of $f(x)$.



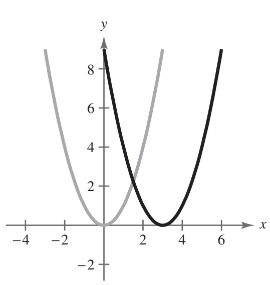
89. $${}_nC_{n-r} = \frac{n!}{(n-(n-r))!(n-r)!}$$
$$= \frac{n!}{r!(n-r)!}$$
$$= \frac{n!}{(n-r)!r!}$$
$$= {}_nC_r$$

90. $0 = (1 - 1)^n = {}_nC_0 - {}_nC_1 + {}_nC_2 - {}_nC_3 + \cdots \pm\ {}_nC_n$

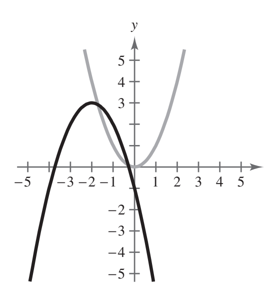
91. $${}_nC_r + {}_nC_{r-1} = \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!}$$
$$= \frac{n!(n-r+1)!(r-1)! + n!(n-r)!r!}{(n-r)!r!(n-r+1)!(r-1)!}$$
$$= \frac{n![(n-r+1)!(r-1)! + r!(n-r)!]}{(n-r)!r!(n-r+1)!(r-1)!}$$
$$= \frac{n!\cancel{(r-1)}![(n-r+1)! + r(n-r)!]}{(n-r)!r!(n-r+1)!\cancel{(r-1)}!}$$
$$= \frac{n!\cancel{(n-r)}![(n-r+1) + r]}{\cancel{(n-r)}!r!(n-r+1)!}$$
$$= \frac{n![n+1]}{r!(n-r+1)!}$$
$$= \frac{(n+1)!}{[(n+1)-r]!r!}$$
$$= {}_{n+1}C_r$$

92. ${}_nC_0 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \cdots + {}_nC_n = (1 + 1)^n = 2^n$

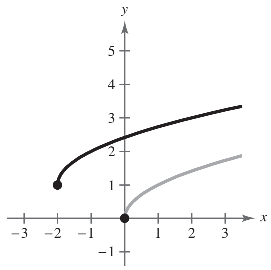
93. The graph of $f(x) = x^2$ is shifted three units to the right. Thus, $g(x) = (x - 3)^2$.



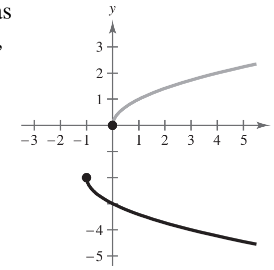
94. The graph of $f(x) = x^2$ has been reflected in the x–axis, shifted two units to the left, and shifted three units upward. Thus, $g(x) = -(x + 2)^2 + 3$.



95. The graph of $f(x) = \sqrt{x}$ is shifted two units to the left and shifted one unit upward. Thus, $g(x) = \sqrt{x + 2} + 1$.



96. The graph of $f(x) = \sqrt{x}$ has been reflected in the x–axis, shifted one unit to the left, and shifted two units downward. Thus, $g(x) = -\sqrt{x + 1} - 2$.



97. $A^{-1} = \dfrac{1}{(-6)(4) - (5)(-5)}\begin{bmatrix} 4 & -5 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 5 & -6 \end{bmatrix}$

98. $[A \ \vdots \ I] = \left[\begin{array}{cc:cc} 1.2 & -2.3 & 1 & 0 \\ -2 & 4 & 0 & 1 \end{array}\right]$

$0.1R_2 + R_1 \rightarrow \left[\begin{array}{cc:cc} 1 & -1.9 & 1 & 0.1 \\ -2 & 4 & 0 & 1 \end{array}\right]$

$2R_1 + R_2 \rightarrow \left[\begin{array}{cc:cc} 1 & -1.9 & 1 & 0.1 \\ 0 & 0.2 & 2 & 1.2 \end{array}\right]$

$5R_2 \rightarrow \left[\begin{array}{cc:cc} 1 & -1.9 & 1 & 0.1 \\ 0 & 1 & 10 & 6 \end{array}\right]$

$1.9R_2 + R_1 \rightarrow \left[\begin{array}{cc:cc} 1 & 0 & 20 & 11.5 \\ 0 & 1 & 10 & 6 \end{array}\right] = [I \ \vdots \ A^{-1}]$

$A^{-1} = \begin{bmatrix} 20 & 11.5 \\ 10 & 6 \end{bmatrix}$

Section 9.6 Counting Principles

- You should know The Fundamental Counting Principle.
- ${}_nP_r = \dfrac{n!}{(n - r)!}$ is the number of permutations of n elements taken r at a time.
- Given a set of n objects that has n_1 of one kind, n_2 of a second kind, and so on, the number of distinguishable permutations is

  $$\frac{n!}{n_1!n_2! \ldots n_k!}.$$
- ${}_nC_r = \dfrac{n!}{(n - r)!r!}$ is the number of combinations of n elements taken r at a time.

Vocabulary Check

1. Fundamental Counting Principle

2. permutation

3. ${}_nP_r = \dfrac{n!}{(n - r)!}$

4. distinguishable permutations

5. combinations

1. Odd integers: 1, 3, 5, 7, 9, 11

6 ways

2. Even integers: 2, 4, 6, 8, 10, 12

6 ways

3. Prime integers: 2, 3, 5, 7, 11

5 ways

4. Greater than 9: 10, 11, 12

3 ways

5. Divisible by 4: 4, 8, 12

3 ways

6. Divisible by 3: 3, 6, 9, 12

4 ways

7. Sum is 9: $1 + 8, 2 + 7, 3 + 6, 4 + 5, 5 + 4,$

$6 + 3, 7 + 2, 8 + 1$

8 ways

8. Two *distinct* integers whose sum is 8:

$1 + 7, 2 + 6, 3 + 5, 5 + 3, 6 + 2, 7 + 1$

6 ways

9. Amplifiers: 3 choices

Compact disc players: 2 choices

Speakers: 5 choices

Total: $3 \cdot 2 \cdot 5 = 30$ ways

10. Chemist: 5 choices

Statistician: 3 choices

Total: $5 \cdot 3 = 15$ ways

11. Math courses: 2

Science courses: 3

Social sciences and
humanities courses: 5

Total: $2 \cdot 3 \cdot 5 = 30$ schedules

12. 1^{st} position: 2

2^{nd} position: 1

3^{rd} position: 6

4^{th} position: 5

5^{th} position: 4

6^{th} position: 3

7^{th} position: 2

8^{th} position: 1

Total: $2!6! = 1440$ ways

13. $2^6 = 64$

14. $2^{12} = 4096$ ways

15. $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175{,}760{,}000$ distinct license plate numbers

16. $24 \cdot 24 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 5{,}760{,}000$ distinct license plates

17. (a) $9 \cdot 10 \cdot 10 = 900$

(b) $9 \cdot 9 \cdot 8 = 648$

(c) $9 \cdot 10 \cdot 2 = 180$

(d) $6 \cdot 10 \cdot 10 = 600$

18. (a) $9 \cdot 10 \cdot 10 \cdot 10 = 9000$ numbers

(b) $9 \cdot 9 \cdot 8 \cdot 7 = 4536$ numbers

(c) $4 \cdot 10 \cdot 10 \cdot 10 = 4000$ numbers

(d) $9 \cdot 10 \cdot 10 \cdot 5 = 4500$ numbers

19. $40^3 = 64{,}000$

20. $50^3 = 125{,}000$ combinations

21. (a) $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$

(b) $8 \cdot 1 \cdot 6 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1 = 384$

22. (a) $8! = 40{,}320$ orders

(b) $4!4! = 576$ orders

23. ${}_nP_r = \dfrac{n!}{(n - r)!}$

So, ${}_4P_4 = \dfrac{4!}{0!} = 4! = 24.$

24. ${}_nP_r = \dfrac{n!}{(n - r)!}$

${}_5P_5 = \dfrac{5!}{(5 - 5)!} = \dfrac{5!}{0!} = 120$

25. ${}_8P_3 = \dfrac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336$

26. ${}_{20}P_2 = \dfrac{20!}{18!} = 20 \cdot 19 = 380$

27. ${}_5P_4 = \dfrac{5!}{1!} = 120$

28. ${}_7P_4 = \dfrac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$

29. $14 \cdot {}_nP_3 = {}_{n+2}P_4$ **Note:** $n \geq 3$ for this to be defined.

$$14\left(\frac{n!}{(n-3)!}\right) = \frac{(n+2)!}{(n-2)!}$$

$$14n(n-1)(n-2) = (n+2)(n+1)n(n-1) \quad \text{(We can divide here by } n(n-1) \text{ since } n \neq 0, n \neq 1.)$$

$$14(n-2) = (n+2)(n+1)$$

$$14n - 28 = n^2 + 3n + 2$$

$$0 = n^2 - 11n + 30$$

$$0 = (n-5)(n-6)$$

$$n = 5 \quad \text{or} \quad n = 6$$

30. ${}_nP_5 = 18 \cdot {}_{n-2}P_4$ **Note:** $n \geq 6$ for this to be defined.

$$\frac{n!}{(n-5)!} = 18\left(\frac{(n-2)!}{(n-6)!}\right)$$

$$n(n-1)(n-2)(n-3)(n-4) = 18(n-2)(n-3)(n-4)(n-5) \quad \left(\begin{array}{l}\text{We can divide by } (n-2), (n-3),\\ (n-4) \text{ since } n \neq 2, n \neq 3, \text{ and } n \neq 4.\end{array}\right)$$

$$n^2 - n = 18n - 90$$

$$n^2 - 19n + 90 = 0$$

$$(n-9)(n-10) = 0$$

$$n = 9 \text{ or } n = 10$$

31. ${}_{20}P_5 = 1{,}860{,}480$

32. ${}_{100}P_5 = 9{,}034{,}502{,}400$

33. ${}_{100}P_3 = 970{,}200$

34. ${}_{10}P_8 = 1{,}814{,}400$

35. ${}_{20}C_5 = 15{,}504$

36. ${}_{10}C_7 = 120$

37. $5! = 120$ ways

38. $6! = 720$ ways

39. ${}_{12}P_4 = \dfrac{12!}{8!} = 12 \cdot 11 \cdot 10 \cdot 9 = 11{,}880$ ways

40. $4! = 24$ orders

41. $\dfrac{7!}{2!1!3!1!} = \dfrac{7!}{2!3!} = 420$

42. $\dfrac{8!}{3!5!} = 56$

43. $\dfrac{7!}{2!1!1!1!1!1!} = \dfrac{7!}{2!} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$

44. $\dfrac{11!}{1!4!4!2!} = \dfrac{11!}{4!4!2!} = 34{,}650$

45.

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

46. ABCD
ACBD
DBCA
DCBA

47. ${}_{15}P_9 = \dfrac{15!}{6!} = 1{,}816{,}214{,}400$
different batting orders

48. ${}_6P_3 = \dfrac{6!}{3!} = 120$

49. ${}_{40}C_{12} = \dfrac{40!}{28!12!} = 5{,}586{,}853{,}480$ ways

50. $_{100}C_{14} = \dfrac{100!}{(100 - 14)!14!}$

$= \dfrac{100!}{86!14!}$

$= 4.42 \times 10^{16}$

51. $_6C_2 = 15$

The 15 ways are listed below.

AB, AC, AD, AE, AF, BC, BD, BE, BF, CD, CE, CF, DE, DF, EF

52. $_{20}C_5 = 15{,}504$ groups

53. $_{35}C_5 = \dfrac{35!}{30!5!} = 324{,}632$ ways

54. $_{40}C_6 = 3{,}838{,}380$ ways

55. There are 7 good units and 3 defective units.

(a) $_7C_4 = \dfrac{7!}{3!4!} = 35$ ways

(b) $_7C_2 \cdot {_3C_2} = \dfrac{7!}{5!2!} \cdot \dfrac{3!}{1!2!} = 21 \cdot 3 = 63$ ways

(c) $_7C_4 + {_7C_3} \cdot {_3C_1} + {_7C_2} \cdot {_3C_2} = \dfrac{7!}{3!4!} + \dfrac{7!}{4!3!} \cdot \dfrac{3!}{2!1!} + \dfrac{7!}{5!2!} \cdot \dfrac{3!}{1!2!}$

$= 35 + 35 \cdot 3 + 21 \cdot 3$

$= 203$ ways

56. (a) $_3C_2 = \dfrac{3!}{2!1!} = 3$ relationships

(b) $_8C_2 = \dfrac{8!}{2!6!} = \dfrac{8 \cdot 7}{2} = 28$ relationships

(c) $_{12}C_2 = \dfrac{12!}{2!10!} = \dfrac{12 \cdot 11}{2} = 66$ relationships

(d) $_{20}C_2 = \dfrac{20!}{2!18!} = \dfrac{20 \cdot 19}{2} = 190$ relationships

57. (a) Select type of card for three of a kind: $_{13}C_1$

Select three of four cards for three of a kind: $_4C_3$

Select type of card for pair: $_{12}C_1$

Select two of four cards for pair: $_4C_2$

$_{13}C_1 \cdot {_4C_3} \cdot {_{12}C_1} \cdot {_4C_2} = \dfrac{13!}{(13 - 1)!1!} \cdot \dfrac{4!}{(4 - 3)!3!} \cdot \dfrac{12!}{(12 - 1)!1!} \cdot \dfrac{4!}{(4 - 2)!2!} = 3744$

(b) Select two jacks: $_4C_2$

Select three aces: $_4C_3$

$_4C_2 \cdot {_4C_3} = \dfrac{4!}{(4 - 2)!2!} \cdot \dfrac{4!}{(4 - 3)!3!} = 24$

58. (a) $_8C_4 = \dfrac{8!}{(8 - 4)!4!} = \dfrac{8!}{4!4!} = \dfrac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 70$ ways

(b) $_3C_2 \cdot {_5C_2} = \dfrac{3!}{(3 - 2)!2!} \cdot \dfrac{5!}{(5 - 2)!2!} = 3 \cdot 10 = 30$ ways

59. $_7C_1 \cdot {_{12}C_3} \cdot {_{20}C_2} = \dfrac{7!}{(7 - 1)!1!} \cdot \dfrac{12!}{(12 - 3)!3!} \cdot \dfrac{20!}{(20 - 2)!2!} = 292{,}600$

60. (a) $(195)(99)(89)(105)(74) \approx 1.335 \times 10^{10}$ different faces

(b) $(89)(105)(74) = 691{,}530$ different faces

61. ${}_5C_2 - 5 = 10 - 5 = 5$ diagonals

62. ${}_6C_2 - 6 = 15 - 6 = 9$ diagonals

63. ${}_8C_2 - 8 = 28 - 8 = 20$ diagonals

64. ${}_{10}C_2 - 10 = 45 - 10 = 35$ diagonals

65. (a) ${}_{53}C_5 \cdot (42) = 120{,}526{,}770$

(b) 1. If the jackpot is won, then there is only one winning number.

(c) There are 22,957,480 possible winning numbers in the state lottery, which is less than the possible number of winning Powerball numbers.

66. (a) Permutation because order matters

(b) Combination because order does not matter

(c) Permutation because order matters

(d) Combination because order does not matter

67. False.
It is an example of a combination.

68. True by the definition of the Fundamental Counting Principle

69. ${}_nC_r = {}_nC_{n-r}$ They are the same.

70. ${}_{10}P_6 > {}_{10}C_6$
Changing the order of any of the six elements selected results in a different permutation but the same combination.

71. $${}_nP_{n-1} = \frac{n!}{(n-(n-1))!} = \frac{n!}{1!} = \frac{n!}{0!} = {}_nP_n$$

72. $${}_nC_n = \frac{n!}{(n-n)!n!} = \frac{n!}{0!n!} = \frac{n!}{n!0!} = \frac{n!}{(n-0)!0!} = {}_nC_0$$

73. $${}_nC_{n-1} = \frac{n!}{(n-(n-1))!(n-1)!} = \frac{n!}{(1)!(n-1)!}$$
$$= \frac{n!}{(n-1)!1!} = {}_nC_1$$

74. $${}_nC_r = \frac{n!}{(n-r)!r!}$$
$$= \frac{n(n-1)(n-2)\cdot\cdot\cdot(n-r+1)(n-r)!}{(n-r)!r!}$$
$$= \frac{n(n-1)(n-2)\cdot\cdot\cdot(n-r+1)}{r!}$$
$$= \frac{{}_nP_r}{r!}$$

75. ${}_{100}P_{80} \approx 3.836 \times 10^{139}$

This number is too large for some calculators to evaluate.

76. The symbol ${}_nP_r$ denotes the number of ways to choose and order r elements out of a collection of n elements.

77. $f(x) = 3x^2 + 8$

(a) $f(3) = 3(3)^2 + 8 = 35$

(b) $f(0) = 3(0)^2 + 8 = 8$

(c) $f(-5) = 3(-5)^2 + 8 = 83$

78. $g(x) = \sqrt{x-3} + 2$

(a) $g(3) = \sqrt{3-3} + 2 = 2$

(b) $g(7) = \sqrt{7-3} + 2 = 4$

(c) $g(x+1) = \sqrt{x+1-3} + 2 = \sqrt{x-2} + 2$

79. $f(x) = -|x-5| + 6$

(a) $f(-5) = -|-5-5| + 6 = -10 + 6 = -4$

(b) $f(-1) = -|-1-5| + 6 = -6 + 6 = 0$

(c) $f(11) = -|11-5| + 6 = -6 + 6 = 0$

80. $f(x) = \begin{cases} x^2 - 2x + 5, & x \le -4 \\ -x^2 - 2, & x > -4 \end{cases}$

(a) $f(-4) = (-4)^2 - 2(-4) + 5 = 29$

(b) $f(-1) = -(-1)^2 - 2 = -3$

(c) $f(-20) = (-20)^2 - 2(-20) + 5 = 445$

81. $\sqrt{x-3} = x - 6$

$(\sqrt{x-3})^2 = (x-6)^2$

$x - 3 = x^2 - 12x + 36$

$0 = x^2 - 13x + 39$

By the Quadratic Formula we have: $x = \dfrac{13 \pm \sqrt{13}}{2}$

$x = \dfrac{13 - \sqrt{13}}{2}$ is extraneous.

The only valid solution is $x = \dfrac{13 + \sqrt{13}}{2} \approx 8.30.$

82. $\dfrac{4}{t} + \dfrac{3}{2t} = 1$

$\dfrac{4}{t}(2t) + \dfrac{3}{2t}(2t) = 1(2t)$

$8 + 3 = 2t$

$5.5 = t$

83. $\log_2(x-3) = 5$

$x - 3 = 2^5$

$x - 3 = 32$

$x = 35$

84. $e^{x/3} = 16$

$\dfrac{x}{3} = \ln 16$

$x = 3 \ln 16 \approx 8.32$

Section 9.7 Probability

You should know the following basic principles of probability.

- If an event E has $n(E)$ equally likely outcomes and its sample space has $n(S)$ equally likely outcomes, then the probability of event E is

  $$P(E) = \frac{n(E)}{n(S)}, \text{ where } 0 \le P(E) \le 1.$$

- If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

  If A and B are not mutually exclusive events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- If A and B are independent events, then the probability that both A and B will occur is $P(A)P(B)$.
- The complement of an event A is denoted by A' and its probability is $P(A') = 1 - P(A)$.

Vocabulary Check

1. experiment; outcomes
2. sample space
3. probability
4. impossible; certain
5. mutually exclusive
6. independent
7. complement
8. (a) iii (b) i (c) iv (d) ii

1. $\{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6),$
$(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

2. $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

3. {ABC, ACB, BAC, BCA, CAB, CBA}

4. {(red, red), (red, blue), (red, yellow), (blue, blue), (blue, yellow)}

5. {AB, AC, AD, AE, BC, BD, BE, CD, CE, DE}

6. {SSS, SSF, SFS, FSS, SFF, FFS, FSF, FFF}

7. $E = \{HHT, HTH, THH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

8. $E = \{HHH, HHT, HTH, HTT\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

9. $E = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{7}{8}$$

10. $E = \{HHH, HHT, HTH, THH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

11. $E = \{K\clubsuit, K\diamondsuit, K\heartsuit, K\spadesuit, Q\clubsuit, Q\diamondsuit, Q\heartsuit, Q\spadesuit, J\clubsuit, J\diamondsuit, J\heartsuit, J\spadesuit\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{12}{52} = \frac{3}{13}$$

12. The probability that the card is *not* a face card is the complement of getting a face card. (See Exercise 11.)

$$P(E') = 1 - P(E) = 1 - \frac{3}{13} = \frac{10}{13}$$

13. $E = \{K\diamondsuit, K\heartsuit, Q\diamondsuit, Q\heartsuit, J\diamondsuit, J\heartsuit\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{52} = \frac{3}{26}$$

14. There are six possible cards in each of 4 suits: $6 \cdot 4 = 24$

$$P(E) = \frac{n(E)}{n(S)} = \frac{24}{52} = \frac{6}{13}$$

15. $E = \{(1, 3), (2, 2), (3, 1)\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

16. $E = \{(1, 6), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 3), (4, 4), (4, 5), (4, 6), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{21}{36} = \frac{7}{12}$$

17. Use the complement.

$$E' = \{(5, 6), (6, 5), (6, 6)\}$$

$$P(E') = \frac{n(E')}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

$$P(E) = 1 - P(E') = 1 - \frac{1}{12} = \frac{11}{12}$$

18. $E = \{(1, 1), (1, 2), (2, 1), (6, 6)\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

19. $E_3 = \{(1, 2), (2, 1)\},\ n(E_3) = 2$

$E_5 = \{(1, 4), (2, 3), (3, 2), (4, 1)\},\ n(E_5) = 4$

$E_7 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\},\ n(E_7) = 6$

$E = E_3 \cup E_5 \cup E_7$

$n(E) = 2 + 4 + 6 = 12$

$$P(E) = \frac{n(E)}{n(S)} = \frac{12}{36} = \frac{1}{3}$$

20. $E = \{(1, 1), (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (3, 6), (4, 1), (4, 3), (4, 5), (5, 2), (5, 4), (5, 6), (6, 1), (6, 3), (6, 5)\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{19}{36}$$

21. $P(E) = \dfrac{{}_3C_2}{{}_6C_2} = \dfrac{3}{15} = \dfrac{1}{5}$

22. $P(E) = \dfrac{{}_2C_2}{{}_6C_2} = \dfrac{1}{15}$

23. $P(E) = \dfrac{{}_4C_2}{{}_6C_2} = \dfrac{6}{15} = \dfrac{2}{5}$

24.
$$P(E) = \frac{{}_1C_1 \cdot {}_2C_1 + {}_1C_1 \cdot {}_3C_1 + {}_2C_1 \cdot {}_3C_1}{{}_6C_2}$$

$$= \frac{2 + 3 + 6}{15} = \frac{11}{15}$$

25. $P(E') = 1 - P(E) = 1 - 0.7 = 0.3$

26. $P(E') = 1 - P(E) = 1 - 0.36 = 0.64$

27. $P(E') = 1 - P(E) = 1 - \frac{1}{4} = \frac{3}{4}$

28. $1 - P(E) = 1 - \frac{2}{3} = \frac{1}{3}$

29. $P(E) = 1 - P(E')$
$= 1 - 0.14 = 0.86$

30. $1 - P(E') = 1 - 0.92 = 0.08$

31. $P(E) = 1 - P(E') = 1 - \frac{17}{35} = \frac{18}{35}$

32. $1 - P(E') = 1 - \frac{61}{100} = \frac{39}{100}$

33. (a) $\frac{290}{500} = 0.58 = 58\%$
(b) $\frac{478}{500} = 0.956 = 95.6\%$
(c) $\frac{2}{500} = 0.004 = 0.4\%$

34. (a) $\frac{34}{100} = 0.34 = 34\%$
(b) $\frac{45}{100} = 0.45 = 45\%$
(c) $\frac{23}{100} = 0.23 = 23\%$

35. (a) $0.24(1011) \approx 243$ adults
(b) $2\% = \dfrac{1}{50}$
(c) $52\% + 12\% = 64\% = \dfrac{16}{25}$

36. (a) $59\% = \dfrac{59}{100}$
(b) $6\% + 11\% = 17\% = \dfrac{17}{100}$
(c) $1 - \dfrac{13}{100} = \dfrac{87}{100}$

37. (a) $\dfrac{672}{1254} = \dfrac{112}{209}$
(b) $\dfrac{582}{1254} = \dfrac{97}{209}$
(c) $\dfrac{672 - 124}{1254} = \dfrac{548}{1254} = \dfrac{274}{627}$

38. (a) $\dfrac{71 + 53}{202} = \dfrac{124}{202} = \dfrac{62}{101}$
(b) $1 - \dfrac{62}{101} = \dfrac{39}{101}$
(c) $\dfrac{24}{202} = \dfrac{12}{101}$

39. $p + p + 2p = 1$
$p = 0.25$
Taylor: $0.50 = \dfrac{1}{2}$
Moore: $0.25 = \dfrac{1}{4}$
Jenkins: $0.25 = \dfrac{1}{4}$

40. $1 - 0.37 - 0.44 = 0.19 = 19\%$

41. (a) $\dfrac{{}_{15}C_{10}}{{}_{20}C_{10}} = \dfrac{3003}{184{,}756} = \dfrac{21}{1292} \approx 0.016$

(b) $\dfrac{{}_{15}C_8 \cdot {}_5C_2}{{}_{20}C_{10}} = \dfrac{64{,}350}{184{,}756} = \dfrac{225}{646} \approx 0.348$

(c) $\dfrac{{}_{15}C_9 \cdot {}_5C_1}{{}_{20}C_{10}} + \dfrac{{}_{15}C_{10}}{{}_{20}C_{10}} = \dfrac{25{,}025 + 3003}{184{,}756} = \dfrac{28{,}028}{184{,}756} = \dfrac{49}{323} \approx 0.152$

42. Total ways to insert paychecks: $5! = 120$ ways

5 correct: 1 way

4 correct: not possible

3 correct: ${}_5C_3 = 10$ ways (because once you choose the three envelopes that will contain the correct paychecks, there is only one way to insert the paychecks so that the other two are wrong)

2 correct: ${}_5C_3 \cdot 2 = 20$ ways (because once you choose the two envelopes that will contain the correct paychecks, there are two ways to fill the next envelope incorrectly, then only one incorrect way to insert the remaining paychecks)

1 correct: $5 \cdot 3 \cdot 3 = 45$ ways (five ways to choose which envelope is paired with the correct paycheck, three ways to fill the next envelope incorrectly, then three ways to fill the envelope whose correct paycheck was placed in the second envelope, and only one way to fill the remaining two envelopes such that both are incorrect)

0 correct: $120 - 1 - 10 - 20 - 45 = 44$ ways

(a) $\dfrac{45}{120} = \dfrac{3}{8}$ (b) $\dfrac{45 + 20 + 10 + 1}{120} = \dfrac{19}{30}$

43. (a) $\dfrac{1}{{}_5P_5} = \dfrac{1}{120}$

(b) $\dfrac{1}{{}_4P_4} = \dfrac{1}{24}$

44. (a) $\dfrac{{}_8C_2 \cdot {}_{100}C_5}{{}_{108}C_7} = \dfrac{\dfrac{8!}{6!2!} \cdot \dfrac{100!}{95!5!}}{\dfrac{108!}{101!7!}} = 0.076$

(b) $\dfrac{{}_8C_2 \cdot {}_{25}C_2 \cdot {}_{25}C_3}{{}_{108}C_7} = \dfrac{\dfrac{8!}{6!2!} \cdot \dfrac{25!}{23!2!} \cdot \dfrac{25!}{22!3!}}{\dfrac{108!}{101!7!}} = 0.00069$

45. (a) $\dfrac{20}{52} = \dfrac{5}{13}$

(b) $\dfrac{26}{52} = \dfrac{1}{2}$

(c) $\dfrac{16}{52} = \dfrac{4}{13}$

46. $\dfrac{{}_{13}C_1 \cdot {}_4C_3 \cdot {}_{12}C_1 \cdot {}_4C_2}{{}_{52}C_5} = \dfrac{13 \cdot 4 \cdot 12 \cdot 6}{2{,}598{,}960}$

$= \dfrac{3744}{2{,}598{,}960}$

$= \dfrac{6}{4165}$

47. (a) $\dfrac{{}_9C_4}{{}_{12}C_4} = \dfrac{126}{495} = \dfrac{14}{55}$ (4 good units)

(b) $\dfrac{{}_9C_2 \cdot {}_3C_2}{{}_{12}C_4} = \dfrac{108}{495} = \dfrac{12}{55}$ (2 good units)

(c) $\dfrac{{}_9C_3 \cdot {}_3C_1}{{}_{12}C_4} = \dfrac{252}{495} = \dfrac{28}{55}$ (3 good units)

At least 2 good units: $\dfrac{12}{55} + \dfrac{28}{55} + \dfrac{14}{55} = \dfrac{54}{55}$

48. (a) $P(EE) = \dfrac{20}{40} \cdot \dfrac{20}{40} = \dfrac{1}{4}$

(b) $P(EO \text{ or } OE) = 2\left(\dfrac{20}{40}\right)\left(\dfrac{20}{40}\right) = \dfrac{1}{2}$

(c) $P(N_1 < 30, N_2 < 30) = \dfrac{29}{40} \cdot \dfrac{29}{40} = \dfrac{841}{1600}$

(d) $P(N_1N_1) = \dfrac{40}{40} \cdot \dfrac{1}{40} = \dfrac{1}{40}$

49. $(0.78)^3 \approx 0.4746$

50. $(0.32)^2 = 0.1024$

51. (a) $P(SS) = (0.985)^2 \approx 0.9702$

(b) $P(S) = 1 - P(FF) = 1 - (0.015)^2 \approx 0.9998$

(c) $P(FF) = (0.015)^2 \approx 0.0002$

52. (a) $P(AA) = (0.90)^2 = 0.81$

(b) $P(NN) = (0.10)^2 = 0.01$

(c) $P(A) = 1 - P(NN) = 1 - 0.01 = 0.99$

53. (a) $P(BBBB) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

(b) $P(BBBB) + P(GGGG) = \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{1}{8}$

(c) $P(\text{at least one boy}) = 1 - P(\text{no boys})$

$= 1 - P(GGGG) = 1 - \frac{1}{16} = \frac{15}{16}$

54. (a) $\frac{1}{38}$

(b) $\frac{18}{38} = \frac{9}{19}$

(c) $\frac{2}{38} + \frac{18}{38} = \frac{20}{38} = \frac{10}{19}$

(d) $\frac{1}{38} \cdot \frac{1}{38} = \frac{1}{1444}$

(e) $\frac{18}{38} \cdot \frac{18}{38} \cdot \frac{18}{38} = \frac{5832}{54{,}872} = \frac{729}{6859}$

(f) a. $\frac{1}{37}$

b. $\frac{18}{37}$

c. $\frac{1}{37} + \frac{18}{37} = \frac{19}{37}$

d. $\frac{1}{37} \cdot \frac{1}{37} = \frac{1}{1369}$

e. $\frac{18}{37} \cdot \frac{18}{37} \cdot \frac{18}{37} = \frac{5832}{50{,}653}$

The probabilities are better for European roulette.

55. $1 - \frac{(45)^2}{(60)^2} = 1 - \left(\frac{45}{60}\right)^2 = 1 - \left(\frac{3}{4}\right)^2 = 1 - \frac{9}{16} = \frac{7}{16}$

56. (a) If the *center* of the coin falls within the circle of radius $d/2$ around a vertex, the coin will cover the vertex.

$$P(\text{coin covers a vertex}) = \frac{\text{Area in which coin may fall so that it covers a vertex}}{\text{Total area}}$$

$$= \frac{n\left[\pi\left(\frac{d}{2}\right)^2\right]}{nd^2} = \frac{\pi}{4}$$

(b) Experimental results will vary.

57. True. Two events are independent if the occurance of one has no effect on the occurance of the other.

58. False. The complement of the event is to roll a number greater than or equal to 3 and its probability is $2/3$.

59. (a) As you consider successive people with distinct birthdays, the probabilities must decrease to take into account the birth dates already used. Because the birth dates of people are independent events, multiply the respective probabilities of distinct birthdays.

(b) $\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365}$

(c) $P_1 = \frac{365}{365} = 1$

$$P_2 = \frac{365}{365} \cdot \frac{364}{365} = \frac{364}{365} P_1 = \frac{365 - (2 - 1)}{365} P_1$$

$$P_3 = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} = \frac{363}{365} P_2 = \frac{365 - (3 - 1)}{365} P_2$$

$$P_n = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365 - (n - 1)}{365} = \frac{365 - (n - 1)}{365} P_{n-1}$$

(d) Q_n is the probability that the birthdays are not distinct which is equivalent to at least two people having the same birthday.

(e)

n	10	15	20	23	30	40	50
P_n	0.88	0.75	0.59	0.49	0.29	0.11	0.03
Q_n	0.12	0.25	0.41	0.51	0.71	0.89	0.97

(f) 23, see the chart above.

60. If a weather forecast indicates that the probability of rain is 40%, this means the meteorological records indicate that over an extended period of time with similar weather conditions it will rain 40% of the time.

61. $6x^2 + 8 = 0$

$6x^2 = -8$

$x^2 = -\frac{4}{3}$

No real solution

62. $4x^2 + 6x - 12 = 0$

$2x^2 + 3x - 6 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4(2)(-6)}}{2(2)}$$

$$= \frac{-3 \pm \sqrt{57}}{4}$$

63. $x^3 - x^2 - 3x = 0$

$x(x^2 - x - 3) = 0$

$x = 0$ or $x^2 - x - 3 = 0$

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-3)}}{2(1)} = \frac{1 \pm \sqrt{13}}{2}$$

64.
$$\begin{aligned} x^5 + x^3 - 2x &= 0 \\ x(x^4 + x^2 - 2) &= 0 \\ x(x^2 + 2)(x^2 - 1) &= 0 \\ x &= 0 \\ x^2 - 1 = 0 &\Longrightarrow x = \pm 1 \end{aligned}$$
$x = 0, \pm 1$

65.
$$\begin{aligned} \frac{12}{x} &= -3 \\ 12 &= -3x \\ -4 &= x \end{aligned}$$

66.
$$\begin{aligned} \frac{32}{x} &= 2x \\ 32 &= 2x^2 \\ 16 &= x^2 \\ \pm 4 &= x \end{aligned}$$

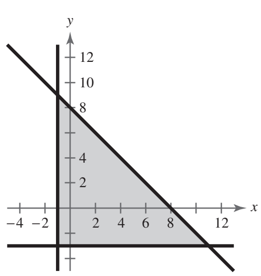
67.
$$\begin{aligned} \frac{2}{x - 5} &= 4 \\ 2 &= 4(x - 5) \\ 2 &= 4x - 20 \\ 22 &= 4x \\ \frac{11}{2} &= x \end{aligned}$$

68.
$$\begin{aligned} \frac{3}{2x + 3} - 4 &= \frac{-1}{2x + 3} \\ \frac{3}{2x + 3} + \frac{1}{2x + 3} &= 4 \\ \frac{4}{2x + 3} &= 4 \\ 4 &= 4(2x + 3) \\ 4 &= 8x + 12 \\ 8x &= -8 \\ x &= -1 \end{aligned}$$

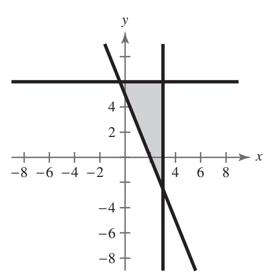
69.
$$\begin{aligned} \frac{3}{x - 2} + \frac{x}{x + 2} &= 1 \\ 3(x + 2) + x(x - 2) &= 1(x - 2)(x + 2) \\ 3x + 6 + x^2 - 2x &= x^2 - 4 \\ x^2 + x + 6 &= x^2 - 4 \\ x + 6 &= -4 \\ x &= -10 \end{aligned}$$

70.
$$\begin{aligned} \frac{2}{x} - \frac{5}{x - 2} &= \frac{-13}{x^2 - 2x} \\ \frac{2(x - 2) - 5x}{x^2 - 2x} &= \frac{-13}{x^2 - 2x} \\ 2x - 4 - 5x &= -13 \\ -4 - 3x &= -13 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

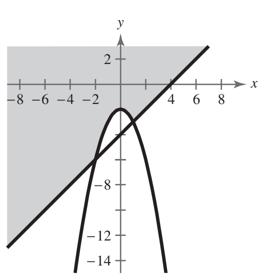
71. $\begin{cases} y \ge -3 \\ x \ge -1 \\ -x - y \ge -8 \end{cases}$



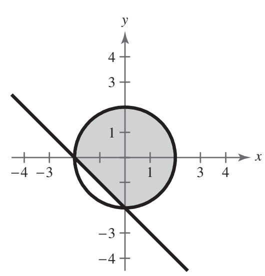
72.



73. $\begin{cases} x^2 + y \ge -2 \\ y \ge x - 4 \end{cases}$



74.



Review Exercises for Chapter 9

1. $a_n = 2 + \frac{6}{n}$

$a_1 = 2 + \frac{6}{1} = 8$

$a_2 = 2 + \frac{6}{2} = 5$

$a_3 = 2 + \frac{6}{3} = 4$

$a_4 = 2 + \frac{6}{4} = \frac{7}{2}$

$a_5 = 2 + \frac{6}{5} = \frac{16}{5}$

2. $a_n = \frac{(-1)^n 5n}{2n - 1}$

$a_1 = \frac{(-1)^1 5(1)}{2(1) - 1} = -5$

$a_2 = \frac{(-1)^2 5(2)}{2(2) - 1} = \frac{10}{3}$

$a_3 = \frac{(-1)^3 5(3)}{2(3) - 1} = -3$

$a_4 = \frac{(-1)^4 5(4)}{2(4) - 1} = \frac{20}{7}$

$a_5 = \frac{(-1)^5 5(5)}{2(5) - 1} = -\frac{25}{9}$

3. $a_n = \frac{72}{n!}$

$a_1 = \frac{72}{1!} = 72$

$a_2 = \frac{72}{2!} = 36$

$a_3 = \frac{72}{3!} = 12$

$a_4 = \frac{72}{4!} = 3$

$a_5 = \frac{72}{5!} = \frac{3}{5}$

4. $a_n = n(n - 1)$

$a_1 = 1(1 - 1) = 0$

$a_2 = 2(2 - 1) = 2$

$a_3 = 3(3 - 1) = 6$

$a_4 = 4(4 - 1) = 12$

$a_5 = 5(5 - 1) = 20$

5. $-2, 2, -2, 2, -2, \ldots$

$a_n = 2(-1)^n$

6. $-1, 2, 7, 14, 23, \ldots$

n:	1	2	3	4	5	$\ldots n$
Terms:	-1	2	7	14	23	$\ldots a_n$

Apparent pattern: Each term is 2 less than the square of n, which implies that $a_n = n^2 - 2$.

7. $4, 2, \frac{4}{3}, 1, \frac{4}{5}, \ldots$

$a_n = \frac{4}{n}$

8. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$

n:	1	2	3	4	5	$\ldots n$
Terms:	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$	$\ldots a_n$

Apparent pattern: Each term is $(-1)^{n+1}$ times the reciprocal of n, which implies that $a_n = \frac{(-1)^{n+1}}{n}$.

9. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

10. $3! \cdot 2! = (3 \cdot 2 \cdot 1) \cdot (2 \cdot 1) = 12$

11. $\frac{3!\,5!}{6!} = \frac{(3 \cdot 2 \cdot 1)5!}{6 \cdot 5!} = 1$

12. $\frac{7! \cdot 6!}{6! \cdot 8!} = \frac{7! \cdot 6!}{6!(8 \cdot 7!)} = \frac{1}{8}$

13. $\sum_{i=1}^{6} 5 = 6(5) = 30$

14. $\sum_{k=2}^{5} 4k = 4(2) + 4(3) + 4(4) + 4(5)$

$= 8 + 12 + 16 + 20 = 56$

15. $\displaystyle\sum_{j=1}^{4} \frac{6}{j^2} = \frac{6}{1^2} + \frac{6}{2^2} + \frac{6}{3^2} + \frac{6}{4^2} = 6 + \frac{3}{2} + \frac{2}{3} + \frac{3}{8} = \frac{205}{24}$

16. $\displaystyle\sum_{i=1}^{8} \frac{i}{i+1} = \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} + \frac{5}{5+1} + \frac{6}{6+1} + \frac{7}{7+1} + \frac{8}{8+1}$

$\displaystyle= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \frac{7}{8} + \frac{8}{9} \approx 6.17$

17. $\displaystyle\sum_{k=1}^{10} 2k^3 = 2(1)^3 + 2(2)^3 + 2(3)^3 + \cdots + 2(10)^3 = 6050$

18. $\displaystyle\sum_{j=0}^{4} (j^2 + 1) = (0^2 + 1) + (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1)$

$= 1 + 2 + 5 + 10 + 17 = 35$

19. $\displaystyle\frac{1}{2(1)} + \frac{1}{2(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{2(20)} = \sum_{k=1}^{20} \frac{1}{2k}$

20. $\displaystyle\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{9}{10} = \sum_{k=1}^{9} \frac{k}{k+1}$

21. $\displaystyle\sum_{i=1}^{\infty} \frac{5}{10^i} = 0.5 + 0.05 + 0.005 + 0.0005 + \cdots = 0.5555\cdots = \frac{5}{9}$

22. $\displaystyle\sum_{i=1}^{\infty} \frac{3}{10^i} = \sum_{i=1}^{\infty} 3\left(\frac{1}{10^i}\right) = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{1}{3}$

23. $\displaystyle\sum_{k=1}^{\infty} \frac{2}{100^k} = 0.02 + 0.0002 + 0.000002 + \cdots = 0.020202\cdots = \frac{2}{99}$

24. $\displaystyle\sum_{k=2}^{\infty} \frac{9}{10^k} = \sum_{k=2}^{\infty} 9\left(\frac{1}{10^k}\right) = \frac{\frac{9}{100}}{1 - \frac{1}{10}} = \frac{1}{10}$

25. $\displaystyle A_n = 10{,}000\left(1 + \frac{0.08}{12}\right)^n$

(a) $A_1 \approx \$10{,}066.67$

$A_2 \approx \$10{,}133.78$

$A_3 \approx \$10{,}201.34$

$A_4 \approx \$10{,}269.35$

$A_5 \approx \$10{,}337.81$

$A_6 \approx \$10{,}406.73$

$A_7 \approx \$10{,}476.10$

$A_8 \approx \$10{,}545.95$

$A_9 \approx \$10{,}616.25$

$A_{10} \approx \$10{,}687.03$

(b) $A_{120} \approx \$22{,}196.40$

26. $a_4 = 734.52$

$a_5 = 750.25$

$a_6 = 768.12$

$a_7 = 788.13$

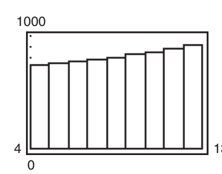
$a_8 = 810.28$

$a_9 = 834.57$

$a_{10} = 861.00$

$a_{11} = 889.57$

$a_{12} = 920.28$



27. $5, 3, 1, -1, -3, \ldots$

Arithmetic sequence, $d = -2$

28. $0, 1, 3, 6, 10, \ldots$

Not an arithmetic sequence

29. $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

Arithmetic sequence, $d = \frac{1}{2}$

30. $\frac{9}{9}, \frac{8}{9}, \frac{7}{9}, \frac{6}{9}, \frac{5}{9}, \ldots$

Arithmetic sequence, $d = -\frac{1}{9}$

31. $a_1 = 4,\ d = 3$

$a_1 = 4$

$a_2 = 4 + 3 = 7$

$a_3 = 7 + 3 = 10$

$a_4 = 10 + 3 = 13$

$a_5 = 13 + 3 = 16$

32. $a_1 = 6, d = -2$

$a_1 = 6$

$a_2 = 6 - 2 = 4$

$a_3 = 4 - 2 = 2$

$a_4 = 2 - 2 = 0$

$a_5 = 0 - 2 = -2$

33. $a_1 = 25,\ a_{k+1} = a_k + 3$

$a_1 = 25$

$a_2 = 25 + 3 = 28$

$a_3 = 28 + 3 = 31$

$a_4 = 31 + 3 = 34$

$a_5 = 34 + 3 = 37$

34. $a_1 = 4.2, a_{k+1} = a_k + 0.4$

$a_1 = 4.2$

$a_2 = 4.2 + 0.4 = 4.6$

$a_3 = 4.6 + 0.4 = 5.0$

$a_4 = 5.0 + 0.4 = 5.4$

$a_5 = 5.4 + 0.4 = 5.8$

35. $a_1 = 7, d = 12$

$$\begin{aligned} a_n &= 7 + (n-1)12 \\ &= 7 + 12n - 12 \\ &= 12n - 5 \end{aligned}$$

36. $a_1 = 25, d = -3$

$a_n = dn + c$

$a_n = -3n + c$

$c = a_1 - d = 25 - (-3) = 28$

So, $a_n = -3n + 28$.

37. $a_1 = y, d = 3y$

$$\begin{aligned} a_n &= y + (n-1)3y \\ &= y + 3ny - 3y \\ &= 3ny - 2y \end{aligned}$$

38. $a_1 = -2x, d = x$

$a_n = dn + c$

$a_n = xn + c$

$c = a_1 - d = -2x - x = -3x$

So, $a_n = xn - 3x$.

39. $a_2 = 93, a_6 = 65$

$a_6 = a_2 + 4d \implies 65 = 93 + 4d \implies -28 = 4d \implies d = -7$

$a_1 = a_2 - d \implies a_1 = 93 - (-7) = 100$

$a_n = a_1 + (n-1)d = 100 + (n-1)(-7) = -7n + 107$

40. $a_7 = 8, a_{13} = 6$

$a_{13} = a_7 + 6d \implies 6 = 8 + 6d \implies d = -\frac{1}{3}$

$a_1 = a_7 - 6d \implies a_1 = 8 - 6\left(-\frac{1}{3}\right) \implies a_1 = 10$

$a_n = a_1 + (n-1)d \implies a_n = 10 + (n-1)\left(-\frac{1}{3}\right) \implies a_n = -\frac{1}{3}n + \frac{31}{3}$

41. $\sum_{j=1}^{10} (2j - 3)$ is arithmetic. Therefore, $a_1 = -1, a_{10} = 17, S_{10} = \frac{10}{2}[-1 + 17] = 80$.

42. $\sum_{j=1}^{8} (20 - 3j) = \sum_{j=1}^{8} 20 - 3\sum_{j=1}^{8} j = 8(20) - 3\left[\frac{(8)(9)}{2}\right] = 52$

43. $\sum_{k=1}^{11} \left(\frac{2}{3}k + 4\right)$ is arithmetic. Therefore, $a_1 = \frac{14}{3}, a_{11} = \frac{34}{3}, S_{11} = \frac{11}{2}\left[\frac{14}{3} + \frac{34}{3}\right] = 88$.

44. $\sum_{k=1}^{25}\left(\frac{3k+1}{4}\right) = \frac{3}{4}\sum_{k=1}^{25} k + \sum_{k=1}^{25} \frac{1}{4} = \frac{3}{4}\left[\frac{(25)(26)}{2}\right] + 25\left(\frac{1}{4}\right) = 250$

45. $\sum_{k=1}^{100} 5k$ is arithmetic. Therefore, $a_1 = 5, a_{100} = 500, S_{500} = \frac{100}{2}(5 + 500) = 25{,}250.$

46. $\sum_{n=20}^{80} n = \sum_{n=1}^{80} n - \sum_{n=1}^{19} n = \frac{(80)(81)}{2} - \frac{(19)(20)}{2} = 3050$

47. $a_n = 34{,}000 + (n - 1)(2250)$

(a) $a_5 = 34{,}000 + 4(2250) = \$43{,}000$

(b) $S_5 = \frac{5}{2}(34{,}000 + 43{,}000) = \$192{,}500$

48. $a_1 = 123, d = 112 - 123 = -11$

$n = 8$

$a_8 = 123 + 7(-11) = 46$

$S_8 = \frac{8}{2}(123 + 46) = 676$

49. $5, 10, 20, 40, \ldots$

The sequence *is* geometric, $r = 2$

50. $54, -18, 6, -2, \ldots$

Geometric sequence, $r = -\frac{18}{54} = -\frac{1}{3}$

51. $\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{8}{3}, \ldots$

The sequence *is* geometric, $r = -2$

52. $\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \ldots$

Not a geometric sequence

53. $a_1 = 4,\ r = -\frac{1}{4}$

$a_1 = 4$

$a_2 = 4\left(-\frac{1}{4}\right) = -1$

$a_3 = -1\left(-\frac{1}{4}\right) = \frac{1}{4}$

$a_4 = \frac{1}{4}\left(-\frac{1}{4}\right) = -\frac{1}{16}$

$a_5 = -\frac{1}{16}\left(-\frac{1}{4}\right) = \frac{1}{64}$

54. $a_1 = 2, r = 2$

$a_1 = 2$

$a_2 = 2(2) = 4$

$a_3 = 4(2) = 8$

$a_4 = 8(2) = 16$

$a_5 = 16(2) = 32$

55. $a_1 = 9,\ a_3 = 4$

$a_3 = a_1 r^2$

$4 = 9r^2$

$\frac{4}{9} = r^2 \implies r = \pm\frac{2}{3}$

$a_1 = 9$		$a_1 = 9$
$a_2 = 9\left(\frac{2}{3}\right) = 6$		$a_2 = 9\left(-\frac{2}{3}\right) = -6$
$a_3 = 6\left(\frac{2}{3}\right) = 4$	or	$a_3 = -6\left(-\frac{2}{3}\right) = 4$
$a_4 = 4\left(\frac{2}{3}\right) = \frac{8}{3}$		$a_4 = 4\left(-\frac{2}{3}\right) = -\frac{8}{3}$
$a_5 = \frac{8}{3}\left(\frac{2}{3}\right) = \frac{16}{9}$		$a_5 = -\frac{8}{3}\left(-\frac{2}{3}\right) = \frac{16}{9}$

56. $a_1 = 2, a_3 = 12$

$a_3 = a_1 r^2$

$12 = 2r^2$

$6 = r^2$

$\pm\sqrt{6} = r$

$a_1 = 2$		$a_1 = 2$
$a_2 = 2\left(\sqrt{6}\right) = 2\sqrt{6}$		$a_2 = 2\left(-\sqrt{6}\right) = -2\sqrt{6}$
$a_3 = 2\sqrt{6}\left(\sqrt{6}\right) = 12$	or	$a_3 = -2\sqrt{6}\left(-\sqrt{6}\right) = 12$
$a_4 = 12\left(\sqrt{6}\right) = 12\sqrt{6}$		$a_4 = 12\left(-\sqrt{6}\right) = -12\sqrt{6}$
$a_5 = 12\sqrt{6}\left(\sqrt{6}\right) = 72$		$a_5 = -12\sqrt{6}\left(-\sqrt{6}\right) = 72$

57. $a_1 = 16, a_2 = -8$

$a_2 = a_1 r \implies -8 = 16r \implies r = -\frac{1}{2}$

$a_n = 16\left(-\frac{1}{2}\right)^{n-1}$

$a_{20} = 16\left(-\frac{1}{2}\right)^{19} \approx -3.052 \times 10^{-5}$

58. $a_3 = 6, a_4 = 1$

$a_3 r = a_4$

$6r = 1$

$r = \frac{1}{6}$

$a_3 = a_1 r^2$

$6 = a_1\left(\frac{1}{6}\right)^2$

$6 = a_1\left(\frac{1}{36}\right)$

$a_1 = 216$

$a_n = 216\left(\frac{1}{6}\right)^{n-1}$

$a_{20} = 216\left(\frac{1}{6}\right)^{19} = 3.545 \times 10^{-13}$

59. $a_1 = 100, r = 1.05$

$a_n = 100(1.05)^{n-1}$

$a_{20} = 100(1.05)^{19} \approx 252.695$

60. $a_1 = 5, r = 0.2$

$a_n = 5(0.2)^{n-1}$

$a_{20} = 5(0.2)^{19} \approx 2.62 \times 10^{-13}$

61. $\sum_{i=1}^{7} 2^{i-1} = \frac{1 - 2^7}{1 - 2} = 127$

62. $\sum_{i=1}^{5} 3^{i-1} = 1\left(\frac{1 - 3^5}{1 - 3}\right) = 121$

63. $\sum_{i=1}^{4} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

64. $\sum_{i=1}^{6} \left(\frac{1}{3}\right)^{i-1} = \left(\frac{1 - \left(\frac{1}{3}\right)^6}{1 - \frac{1}{3}}\right) = \frac{1 - \frac{1}{729}}{1 - \frac{1}{3}} = \frac{364}{243}$

65. $\sum_{i=1}^{5} (2)^{i-1} = 1 + 2 + 4 + 8 + 16 = 31$

66. $\sum_{i=1}^{4} 6(3)^i = 6(3)\left(\frac{1 - 3^4}{1 - 3}\right) = 720$

67. $\sum_{i=1}^{10} 10\left(\frac{3}{5}\right)^{i-1} \approx 24.85$

68. $\sum_{i=1}^{15} 20(0.2)^{i-1} = 25$

69. $\sum_{i=1}^{25} 100(1.06)^{i-1} \approx 5486.45$

70. $\sum_{i=1}^{20} 8\left(\frac{6}{5}\right)^{i-1} = 1493.50$

71. $\sum_{i=1}^{\infty} \left(\frac{7}{8}\right)^{i-1} = \frac{1}{1 - \frac{7}{8}} = 8$

72. $\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$

73. $\sum_{i=1}^{\infty} (0.1)^{i-1} = \frac{1}{1 - 0.1} = \frac{10}{9}$

74. $\sum_{i=1}^{\infty} (0.5)^{i-1} = \frac{1}{1 - 0.5} = 2$

75. $\sum_{k=1}^{\infty} 4\left(\frac{2}{3}\right)^{k-1} = \frac{4}{1 - \frac{2}{3}} = 12$

76. $\sum_{k=1}^{\infty} 1.3\left(\frac{1}{10}\right)^{k-1} = \frac{1.3}{1 - \frac{1}{10}} = \frac{13}{9}$

77. (a) $a_t = 120{,}000(0.7)^t$

(b) $a_5 = 120{,}000(0.7)^5$

$= \$20{,}168.40$

78. Monthly: $A = P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right)$

$= 200\left[\left(1 + \frac{0.06}{12}\right)^{12 \cdot 10} - 1\right]\left(1 + \frac{12}{0.06}\right)$

$= \$32{,}939.75$

Continuously: $A = \frac{Pe^{r/12}(e^{rt} - 1)}{e^{r/12} - 1}$

$= \frac{200e^{0.06/12}(e^{(0.06)(10)} - 1)}{e^{0.06/12} - 1} = \$32{,}967.03$

79. 1. When $n = 1, 3 = 1(1 + 2)$.

2. Assume that $S_k = 3 + 5 + 7 + \cdots + (2k + 1) = k(k + 2)$.

Then,
$$\begin{aligned} S_{k+1} &= 3 + 5 + 7 + \cdots + (2k + 1) + [2(k + 1) + 1] = S_k + (2k + 3) \\ &= k(k + 2) + 2k + 3 \\ &= k^2 + 4k + 3 \\ &= (k + 1)(k + 3) \\ &= (k + 1)[(k + 1) + 2]. \end{aligned}$$

Therefore, by mathematical induction, the formula is valid for all positive integer values of n.

80. 1. When $n = 1, S_1 = 1 = \frac{1}{4}(1 + 3) = 1$.

2. Assume that $S_k = 1 + \frac{3}{2} + 2 + \frac{5}{2} + \cdots + \frac{1}{2}(k + 1) = \frac{k}{4}(k + 3)$. Then,

$$\begin{aligned} S_{k+1} = S_k + a_{k+1} &= \left(1 + \frac{3}{2} + 2 + \frac{5}{2} + \cdots + \frac{1}{2}(k + 1)\right) + \frac{1}{2}(k + 2) \\ &= \frac{k}{4}(k + 3) + \frac{1}{2}(k + 2) \\ &= \frac{k(k + 3) + 2(k + 2)}{4} \\ &= \frac{k^2 + 5k + 4}{4} \\ &= \frac{(k + 1)(k + 4)}{4} \\ &= \frac{k + 1}{4}[(k + 1) + 3]. \end{aligned}$$

Thus, the formula holds for all positive integers n.

81. 1. When $n = 1, a = a\left(\frac{1 - r}{1 - r}\right)$.

2. Assume that $S_k = \sum_{i=0}^{k-1} ar^i = \frac{a(1 - r^k)}{1 - r}$.

Then,
$$\begin{aligned} S_{k+1} &= \sum_{i=0}^{k} ar^i = \left(\sum_{i=0}^{k-1} ar^i\right) + ar^k = \frac{a(1 - r^k)}{1 - r} + ar^k \\ &= \frac{a(1 - r^k + r^k - r^{k+1})}{1 - r} = \frac{a(1 - r^{k+1})}{1 - r}. \end{aligned}$$

Therefore, by mathematical induction, the formula is valid for all positive integer values of n.

82. 1. When $n = 1$, $S_1 = a + 0 \cdot d = a = \frac{1}{2}[2a + (1 - 1)d] = a$.

2. Assume that $S_k = \sum_{k=0}^{i-1}(a + kd) = \frac{i}{2}[2a + (i - 1)d]$. Then,

$$S_{k+1} = S_k + a_{k+1}$$

$$\sum_{k=0}^{i+1-1}(a + kd) = \frac{i}{2}[2a + (i - 1)d] + [a + id]$$

$$= \frac{2ia + i(i - 1)d + 2a + 2id}{2} = \frac{2a(i + 1) + id(i + 1)}{2} = \left(\frac{i + 1}{2}\right)[2a + id].$$

Thus, the formula holds for all positive integers n.

83. $S_1 = 9 = 1(9) = 1[2(1) + 7]$

$S_2 = 9 + 13 = 22 = 2(11) = 2[2(2) + 7]$

$S_3 = 9 + 13 + 17 = 39 = 3(13) = 3[2(3) + 7]$

$S_4 = 9 + 13 + 17 + 21 = 60 = 4(15) = 4[2(4) + 7]$

$S_n = n(2n + 7)$

84. $S_1 = 68 = 4 \cdot 17$

$S_2 = 68 + 60 = 128 = 8 \cdot 16$

$S_3 = 68 + 60 + 52 = 180 = 12 \cdot 15$

$S_4 = 68 + 60 + 52 + 44 = 224 = 16 \cdot 14$

$S_n = 4n(18 - n)$

85. $S_1 = 1$

$S_2 = 1 + \frac{3}{5} = \frac{8}{5}$

$S_3 = 1 + \frac{3}{5} + \frac{9}{25} = \frac{49}{25}$

$S_4 = 1 + \frac{3}{5} + \frac{9}{25} + \frac{27}{125} = \frac{272}{125}$

Since the series is geometric,

$$S_n = \frac{1 - \left(\frac{3}{5}\right)^n}{1 - \frac{3}{5}} = \frac{5}{2}\left[1 - \left(\frac{3}{5}\right)^n\right].$$

86. $S_1 = 12$

$S_2 = 12 - 1 = 11$

$S_3 = 12 - 1 + \frac{1}{12} = \frac{133}{12}$

$S_4 = 12 - 1 + \frac{1}{12} - \frac{1}{144} = \frac{1595}{144}$

Since the series is geometric,

$$S_n = 12\frac{1 - \left(-\frac{1}{12}\right)^n}{\left[1 - \left(-\frac{1}{12}\right)\right]} = \frac{144}{13}\left[1 - \left(-\frac{1}{12}\right)^n\right].$$

87. $\sum_{n=1}^{30} n = \frac{30(31)}{2} = 465$

88. $\sum_{n=1}^{10} n^2 = \frac{10(10 + 1)(2 \cdot 10 + 1)}{6} = \frac{10(11)(21)}{6} = 385$

89. $\sum_{n=1}^{7}(n^4 - n) = \sum_{n=1}^{7} n^4 - \sum_{n=1}^{7} n = \frac{(7)(8)(15)[(3)(49) + 21 - 1]}{30} - \frac{(7)(8)}{2}$

$$= \frac{(7)(8)(15)(167)}{30} - \frac{(7)(8)}{2}$$

$$= 4676 - 28 = 4648$$

90. $\sum_{n=1}^{6}(n^5 - n^2) = \sum_{n=1}^{6} n^5 - \sum_{n=1}^{6} n^2$

$$= \frac{(6)^2(6 + 1)^2[2(6)^2 + 2(6) - 1]}{12} - \frac{6(6 + 1)[2(6) + 1]}{6}$$

$$= \frac{(6)^2(7)^2(83)}{12} - \frac{6(7)(13)}{6}$$

$$= \frac{(6)^2(7)^2(83) - 2(6)(7)(13)}{12} = 12{,}110$$

91. $a_1 = f(1) = 5, \quad a_n = a_{n-1} + 5$

$a_1 = 5$

$a_2 = 5 + 5 = 10$

$a_3 = 10 + 5 = 15$

$a_4 = 15 + 5 = 20$

$a_5 = 20 + 5 = 25$

$$\begin{array}{rccccccccc} n: & 1 & & 2 & & 3 & & 4 & & 5 \\ a_n: & 5 & & 10 & & 15 & & 20 & & 25 \\ \text{First differences:} & & 5 & & 5 & & 5 & & 5 & \\ \text{Second differences:} & & & 0 & & 0 & & 0 & & \end{array}$$

The sequence has a linear model.

92. $a_1 = -3$

$a_n = a_{n-1} - 2n$

$a_1 = -3$

$a_2 = a_1 - 2(2) = -3 - 4 = -7$

$a_3 = a_2 - 2(3) = -7 - 6 = -13$

$a_4 = a_3 - 2(4) = -13 - 8 = -21$

$a_5 = a_4 - 2(5) = -21 - 10 = -31$

$$\begin{array}{rccccccccc} a_n: & -3 & & -7 & & -13 & & -21 & & -31 \\ \text{First differences:} & & -4 & & -6 & & -8 & & -10 & \\ \text{Second differences:} & & & -2 & & -2 & & -2 & & \end{array}$$

Since the second differences are all the same, the sequence has a quadratic model.

93. $a_1 = f(1) = 16, \quad a_n = a_{n-1} - 1$

$a_1 = 16$

$a_2 = 16 - 1 = 15$

$a_3 = 15 - 1 = 14$

$a_4 = 14 - 1 = 13$

$a_5 = 13 - 1 = 12$

$$\begin{array}{rccccccccc} n: & 1 & & 2 & & 3 & & 4 & & 5 \\ a_n: & 16 & & 15 & & 14 & & 13 & & 12 \\ \text{First differences:} & & -1 & & -1 & & -1 & & -1 & \\ \text{Second differences:} & & & 0 & & 0 & & 0 & & \end{array}$$

The sequence has a linear model.

94. $a_0 = 0, a_n = n - a_{n-1}$

$a_0 = 0$

$a_1 = 1 - a_0 = 1 - 0 = 1$

$a_2 = 2 - a_1 = 2 - 1 = 1$

$a_3 = 3 - a_2 = 3 - 1 = 2$

$a_4 = 4 - a_3 = 4 - 2 = 2$

$$\begin{array}{rccccccccc} a_n: & 0 & & 1 & & 1 & & 2 & & 2 \\ \text{First differences:} & & 1 & & 0 & & 1 & & 0 & \\ \text{Second differences:} & & & -1 & & 1 & & -1 & & \end{array}$$

Since neither the first differences nor the second differences are equal, the sequence does not have a linear or a quadratic model.

95. ${}_6C_4 = \dfrac{6!}{2!4!} = 15$

96. ${}_{10}C_7 = \dfrac{10!}{7!3!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7!}{7!3!}$

$= \dfrac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$

97. ${}_8C_5 = \dfrac{8!}{3!5!} = 56$

98. ${}_{12}C_3 = \dfrac{12!}{3!9!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9!}{3! \cdot 9!} = \dfrac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 220$

99. $\dbinom{7}{3} = 35$

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\ 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \end{array}$$

$\dbinom{7}{3} = 35$, the 5^{th} entry in the 7^{th} row

100.

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\ 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\ 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\ 1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \end{array}$$

$\dbinom{9}{4} = 126$, the 5^{th} entry in the 9^{th} row

101. $\binom{8}{6} = 28$

$$\begin{array}{c}1\\1 \quad 1\\1 \quad 2 \quad 1\\1 \quad 3 \quad 3 \quad 1\\1 \quad 4 \quad 6 \quad 4 \quad 1\\1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1\\1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1\\1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1\\1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1\end{array}$$

$\binom{8}{6} = 28$, the 7th entry in the 8th row

102.

$$\begin{array}{c}1\\1 \quad 1\\1 \quad 2 \quad 1\\1 \quad 3 \quad 3 \quad 1\\1 \quad 4 \quad 6 \quad 4 \quad 1\\1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1\end{array}$$

$\binom{5}{3} = 10$, the 4th entry in the 5th row

103. $(x + 4)^4 = x^4 + 4x^3(4) + 6x^2(4)^2 + 4x(4)^3 + 4^4$

$= x^4 + 16x^3 + 96x^2 + 256x + 256$

104. $(x - 3)^6 = {}_6C_0(x)^6(-3)^0 + {}_6C_1(x)^5(-3) + {}_6C_2(x)^4(-3)^2 + {}_6C_3(x)^3(-3)^3 + {}_6C_4(x)^2(-3)^4 + {}_6C_5(x)(-3)^5 + {}_6C_6(x)^0(-3)^6$

$= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729$

105. $(a - 3b)^5 = a^5 - 5a^4(3b) + 10a^3(3b)^2 - 10a^2(3b)^3 + 5a(3b)^4 - (3b)^5$

$= a^5 - 15a^4b + 90a^3b^2 - 270a^2b^3 + 405ab^4 - 243b^5$

106. $(3x + y^2)^7 = {}_7C_0(3x)^7 + {}_7C_1(3x)^6(y^2) + {}_7C_2(3x)^5(y^2)^2 + {}_7C_3(3x)^4(y^2)^3 + {}_7C_4(3x)^3(y^2)^4 + {}_7C_5(3x)^2(y^2)^5$

$+ {}_7C_6(3x)(y^2)^6 + {}_7C_7(y^2)^7$

$= (3x)^7 + 7(3x)^6y^2 + 21(3x)^5(y^2)^2 + 35(3x)^4(y^2)^3 + 35(3x)^3(y^2)^4 + 21(3x)^2(y^2)^5 + 7(3x)(y^2)^6 + (y^2)^7$

$= 2187x^7 + 5103x^6y^2 + 5103x^5y^4 + 2835x^4y^6 + 945x^3y^8 + 189x^2y^{10} + 21xy^{12} + y^{14}$

107. $(5 + 2i)^4 = (5)^4 + 4(5)^3(2i) + 6(5)^2(2i)^2 + 4(5)(2i)^3 + (2i)^4$

$= 625 + 1000i + 600i^2 + 160i^3 + 16i^4$

$= 625 + 1000i - 600 - 160i + 16 = 41 + 840i$

108. $(4 - 5i)^3 = {}_3C_0(4^3) + {}_3C_1(4^2)(-5i) + {}_3C_2(4)(-5i)^2 + {}_3C_3(-5i)^3$

$= 4^3 - 3(4)^2(5i) + 3(4)(5i)^2 - (5i)^3$

$= 64 - 240i - 300 + 125i$

$= -236 - 115i$

109.

First number:	1	2	3	4	5	6	7	8	9	10	11
Second number:	11	10	9	8	7	6	5	4	3	2	1

From this list, you can see that a total of 12 occurs 11 different ways.

110. ${}_6C_1 \cdot {}_5C_1 \cdot {}_6C_1 = 6 \cdot 5 \cdot 6 = 180$

111. $(10)(10)(10)(10) = 10{,}000$ different telephone numbers

112. ${}_3C_1 \cdot {}_4C_1 \cdot {}_6C_1 = 3 \cdot 4 \cdot 6 = 72$

113. ${}_{10}P_3 = \dfrac{10!}{7!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7!}{7!}$

$= 10 \cdot 9 \cdot 8 = 720$ different ways

114. ${}_{32}C_{12} = \dfrac{32!}{20!12!} = 225{,}792{,}840$

115. ${}_8C_3 = \dfrac{8!}{5!3!} = 56$

116. Breads: ${}_5C_1 = 5$

Meats: ${}_7C_0 + {}_7C_1 + {}_7C_2 + {}_7C_3 + {}_7C_4 + {}_7C_5 + {}_7C_6 + {}_7C_7 = 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128$

Cheese: ${}_3C_0 + {}_3C_1 + {}_3C_2 + {}_3C_3 = 1 + 3 + 3 + 1 = 8$

Vegetables: ${}_6C_0 + {}_6C_1 + {}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5 + {}_6C_6 = 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$

$5 \cdot 128 \cdot 8 \cdot 64 = 327{,}680$

117. $(1)\left(\dfrac{1}{9}\right) = \dfrac{1}{9}$

118. $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{1}{5!} = \dfrac{1}{120}$

119. (a) $25\% + 18\% = 43\%$

(b) $100\% - 18\% = 82\%$

120. (a) $\frac{208}{500} = 0.416$ or 41.6%

(b) $\frac{400}{500} = 0.8$ or 80%

(c) $\frac{37}{500} = 0.074$ or 7.4%

121. $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{216}$

122. $\left(\dfrac{6}{6}\right)\left(\dfrac{5}{6}\right)\left(\dfrac{4}{6}\right)\left(\dfrac{3}{6}\right)\left(\dfrac{2}{6}\right)\left(\dfrac{1}{6}\right) = \dfrac{6!}{6^6} = \dfrac{720}{46{,}656} = \dfrac{5}{324}$

123. $1 - \dfrac{13}{52} = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

124. $1 - P(HHHHH) = 1 - \left(\dfrac{1}{2}\right)^5 = \dfrac{31}{32}$

125. True. $\dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$

126. True by Properties of Sums

127. True. $\displaystyle\sum_{k=1}^{8} 3k = 3\sum_{k=1}^{8} k$ by the Properties of Sums.

128. True because $2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2^{3-2} + 2^{4-2} + 2^{5-2} + 2^{6-2} + 2^{7-2} + 2^{8-2}$

129. False. If $r = 0$ or $r = 1$, then ${}_nP_r = {}_nC_r$.

130. The domain of an infinite sequence is the set of natural numbers.

131. (a) Odd-numbered terms are negative.

(b) Even-numbered terms are negative.

132. (a) Arithmetic. There is a constant difference between consecutive terms.

(b) Geometric. Each term is a constant multiple of the previous term. In this case the common ratio is greater than 1.

133. Each term of the sequence is defined in terms of preceding terms.

134. Increased powers of real numbers between 0 and 1 approach zero.

135. $a_n = 4\left(\frac{1}{2}\right)^{n-1}$

$a_1 = 4, a_2 = 2, a_{10} = \frac{1}{128}$

The sequence is geometric and is decreasing.

Matches graph (d).

136. $a_n = 4\left(-\frac{1}{2}\right)^{n-1}$

$a_1 = 4$ and a_n fluctuates from positive to negative.

Matches graph (a).

137. $a_n = \displaystyle\sum_{k=1}^{n} 4\left(\tfrac{1}{2}\right)^{k-1}$

$a_1 = 4$ and $a_n \longrightarrow 8$ as $n \longrightarrow \infty$

Matches graph (b).

138. $a_n = \sum_{k=1}^{n} 4\left(-\frac{1}{2}\right)^{k-1}$

$a_1 = 4$ and $a_n \to \frac{8}{3}$ as $n \to \infty$.

Matches graph (c).

139. $S_6 = S_5 + S_4 + S_3 = 130 + 70 + 40 = 240$

$S_7 = S_6 + S_5 + S_4 = 240 + 130 + 70 = 440$

$S_8 = S_7 + S_6 + S_5 = 440 + 240 + 130 = 810$

$S_9 = S_8 + S_7 + S_6 = 810 + 440 + 240 = 1490$

$S_{10} = S_9 + S_8 + S_7 = 1490 + 810 + 440 = 2740$

140. $0 \le p \le 1$, closed interval

Problem Solving for Chapter 9

1. $x_0 = 1$ and $x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, n = 1, 2, \ldots$

$x_0 = 1$

$x_1 = \frac{1}{2}(1) + \frac{1}{1} = \frac{3}{2} = 1.5$

$x_2 = \frac{1}{2}\left(\frac{3}{2}\right) + \frac{1}{3/2} = \frac{17}{12} = 1.41\overline{6}$

$x_3 = \frac{1}{2}\left(\frac{17}{12}\right) + \frac{1}{17/12} = \frac{577}{408} \approx 1.414215686$

$x_4 = \frac{1}{2}\left(\frac{577}{408}\right) + \frac{1}{577/408} \approx 1.414213562$

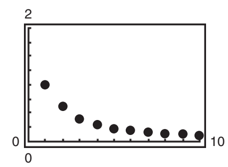
$x_5 = \frac{1}{2}x_4 + \frac{1}{x_4} \approx 1.414213562$

$x_6 \approx x_7 \approx x_8 \approx x_9 \approx 1.414213562$

Conjecture: $x_n \to \sqrt{2}$ as $n \to \infty$

2. $a_n = \frac{n+1}{n^2+1}$

(a)



(b) $a_n \to 0$ as $n \to \infty$

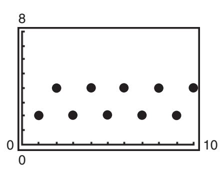
(c)

n	1	10	100	1000	10,000
a_n	1	$\frac{11}{101}$	$\frac{101}{10,001}$	$\frac{1001}{1,000,001}$	$\frac{10,001}{100,000,001}$

(d) $a_n \to 0$ as $n \to \infty$

3. $a_n = 3 + (-1)^n$

(a)



(b) $a_n = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 4, & \text{if } n \text{ is even} \end{cases}$

(c)

n	1	10	101	1000	10,001
a_n	2	4	2	4	2

(d) As $n \to \infty$, a_n oscillates between 2 and 4 and does not approach a fixed value.

4. Let $a_n = dn + c$, an arithmetic sequence with a common difference of d.

(a) If C is added to each term, then the resulting sequence, $b_n = a_n + C = dn + c + C$ is still arithmetic with a common difference of d.

(b) If each term is multiplied by a nonzero constant C, then the resulting sequence, $b_n = C(dn + c) = Cdn + Cc$ is still arithmetic. The common difference is Cd.

(c) If each term is squared, the resulting sequence, $b_n = a_n{}^2 = (dn + c)^2$ is not arithmetic.

5. (a)

$$\begin{array}{lccccccccccccccccc} & 1 & & 4 & & 9 & & 16 & & 25 & & 36 & & 49 & & 64 & & 81 \\ \text{First differences:} & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & 15 & & 17 & \end{array}$$

In general, $b_n = 2n + 1$ for the first differences.

(b) Find the second differences of the perfect cubes.

(c)

$$\begin{array}{lccccccccccccccccc} & 1 & & 8 & & 27 & & 64 & & 125 & & 216 & & 343 & & 512 & & 729 \\ \text{First differences:} & & 7 & & 19 & & 37 & & 61 & & 91 & & 127 & & 169 & & 217 & \\ \text{Second differences:} & & & 12 & & 18 & & 24 & & 30 & & 36 & & 42 & & 48 & & \end{array}$$

In general, $c_n = 6(n + 1) = 6n + 6$ for the second differences.

(d) Find the third differences of the perfect fourth powers.

(e)

$$\begin{array}{lccccccccccccccccc} & 1 & & 16 & & 81 & & 256 & & 625 & & 1296 & & 2401 & & 4096 & & 6561 \\ \text{First differences:} & & 15 & & 65 & & 175 & & 369 & & 671 & & 1105 & & 1695 & & 2465 & \\ \text{Second differences:} & & & 50 & & 110 & & 194 & & 302 & & 434 & & 590 & & 770 & & \\ \text{Third differences:} & & & & 60 & & 84 & & 108 & & 132 & & 156 & & 180 & & & \end{array}$$

In general, $d_n = 24n + 36$ for the third differences.

6. Distance: $\displaystyle\sum_{n=1}^{\infty} 20\left(\frac{1}{2}\right)^{n-1} = \frac{20}{1 - \frac{1}{2}} = 40$

Time: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1 - \frac{1}{2}} = 2$

In two seconds, both Achilles and the tortoise will be 40 feet away from Achilles starting point.

7. Side lengths: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$$S_n = \left(\frac{1}{2}\right)^{n-1} \text{ for } n \geq 1$$

Areas: $\dfrac{\sqrt{3}}{4}, \dfrac{\sqrt{3}}{4}\left(\dfrac{1}{2}\right)^2, \dfrac{\sqrt{3}}{4}\left(\dfrac{1}{4}\right)^2, \dfrac{\sqrt{3}}{4}\left(\dfrac{1}{8}\right)^2, \ldots$

$$A_n = \frac{\sqrt{3}}{4}\left[\left(\frac{1}{2}\right)^{n-1}\right]^2 = \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^{2n-2} = \frac{\sqrt{3}}{4}S_n^{\,2}$$

8. $a_n = \begin{cases} \dfrac{a_{n-1}}{2}, & \text{if } a_{n-1} \text{ is even} \\ 3a_{n-1} + 1, & \text{if } a_{n-1} \text{ is odd} \end{cases}$

(a)

$a_1 = 7$

$a_2 = 3(7) + 1 = 22$

$a_3 = \frac{22}{2} = 11$

$a_4 = 3(11) + 1 = 34$

$a_5 = \frac{34}{2} = 17$

$a_6 = 3(17) + 1 = 52$

$a_7 = \frac{52}{2} = 26$

$a_8 = \frac{26}{2} = 13$

$a_9 = 3(13) + 1 = 40$

$a_{10} = \frac{40}{2} = 20$

$a_{11} = \frac{20}{2} = 10$

$a_{12} = \frac{10}{2} = 5$

$a_{13} = 3(5) + 1 = 16$

$a_{14} = \frac{16}{2} = 8$

$a_{15} = \frac{8}{2} = 4$

$a_{16} = \frac{4}{2} = 2$

$a_{17} = \frac{2}{2} = 1$

$a_{18} = 3(1) + 1 = 4$

$a_{19} = \frac{4}{2} = 2$

$a_{20} = \frac{2}{2} = 1$

(b)

$a_1 = 4$	$a_1 = 5$	$a_1 = -3$
$a_2 = 2$	$a_2 = 16$	$a_2 = -8$
$a_3 = 1$	$a_3 = 8$	$a_3 = -4$
$a_4 = 4$	$a_4 = 4$	$a_4 = -2$
$a_5 = 2$	$a_5 = 2$	$a_5 = -1$
$a_6 = 1$	$a_6 = 1$	$a_6 = -2$
$a_7 = 4$	$a_7 = 4$	$a_7 = -1$
$a_8 = 2$	$a_8 = 2$	$a_8 = -2$
$a_9 = 1$	$a_9 = 1$	$a_9 = -1$
$a_{10} = 4$	$a_{10} = 4$	$a_{10} = -2$

Eventually the terms repeat; 4, 2, 1 if a_1 is a positive integer and $-2, -1$ if a_1 is a negative integer.

9. The numbers 1, 5, 12, 22, 35, 51, . . . can be written recursively as $P_n = P_{n-1} + (3n - 2)$. Show that $P_n = n(3n - 1)/2$.

1. For $n = 1$: $1 = \dfrac{1(3 - 1)}{2}$

2. Assume $P_k = \dfrac{k(3k - 1)}{2}$.

   Then,
$$\begin{aligned} P_{k+1} &= P_k + [3(k + 1) - 2] \\ &= \frac{k(3k - 1)}{2} + (3k + 1) = \frac{k(3k - 1) + 2(3k + 1)}{2} \\ &= \frac{3k^2 + 5k + 2}{2} = \frac{(k + 1)(3k + 2)}{2} \\ &= \frac{(k + 1)[3(k + 1) - 1]}{2}. \end{aligned}$$

Therefore, by mathematical induction, the formula is valid for all integers $n \geq 1$.

10. (a) If P_3 is true and P_k implies P_{k+1}, then P_n is true for integers $n \geq 3$.

(b) If $P_1, P_2, P_3, \ldots, P_{50}$ are all true, then you can draw *no* conclusion about P_n in general other than it is true for $1 \leq n \leq 50$.

(c) If P_1, P_2, and P_3 are all true, but the truth of P_k does not imply that P_{k+1} is true, then P_n is false for some values of $n \geq 4$. You can only conclude that it is true for P_1, P_2, and P_3.

(d) If P_2 is true and P_{2k} implies P_{2k+2}, then P_{2n} is true for all integers $n \geq 1$.

11. (a) The Fibonacci sequence is defined as follows: $f_1 = 1, f_2 = 1, f_n = f_{n-2} + f_{n-1}$ for $n \geq 3$.

By this definition $f_3 = f_1 + f_2 = 2, f_4 = f_2 + f_3 = 3, f_5 = f_4 + f_3 = 5, f_6 = f_5 + f_4 = 8, \ldots$

1. For $n = 2$: $f_1 + f_2 = 2$ and $f_4 - 1 = 2$

2. Assume $f_1 + f_2 + \ldots + f_k = f_{k+2} - 1$.

   Then, $f_1 + f_2 + f_3 + \ldots + f_k + f_{k+1} = f_{k+2} - 1 + f_{k+1} = (f_{k+2} + f_{k+1}) - 1 = f_{k+3} - 1 = f_{(k+1)+2} - 1$.

Therefore, by mathematical induction, the formula is valid for all integers $n \geq 2$.

(b) $S_{20} = f_{22} - 1 = 17{,}711 - 1 = 17{,}710$

12. (a) Odds against choosing a red marble $= \dfrac{\text{number of non-red marbles}}{\text{number of red marbles}}$

$$\frac{4}{1} = \frac{x}{6}$$

$$24 = x \quad \text{(number of non-red marbles)}$$

$$\text{Total marbles} = 6 + 24 = 30$$

(b) Odds in favor of choosing a blue marble $= \dfrac{\text{number of blue marbles}}{\text{number of yellow marbles}} = \dfrac{3}{7}$

Odds against choosing a blue marble $= \dfrac{\text{number of yellow marbles}}{\text{number of blue marbles}} = \dfrac{7}{3}$

(c) $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{n(E)}{n(E) + n(E')} = \dfrac{n(E)/n(E')}{n(E)/n(E') + n(E')/n(E')}$

$P(E) = \dfrac{\text{odds in favor of } E}{\text{odds in favor of } E + 1}$

—CONTINUED—

12. —CONTINUED—

(d) $P(E) = \dfrac{n(E)}{n(S)}$ $\qquad P(E') = \dfrac{n(E')}{n(S)}$

$n(S)P(E) = n(E) \qquad n(S)P(E') = n(E')$

Odds in favor of event $E = \dfrac{n(E)}{n(E')} = \dfrac{n(S)P(E)}{n(S)P(E')} = \dfrac{P(E)}{P(E')}$

13. $\dfrac{1}{3}$

14. $1 - \dfrac{\text{Area of triangle}}{\text{Area of circle}} = 1 - \dfrac{\frac{1}{2}(12)(6)}{\pi(6)^2} = 1 - \dfrac{1}{\pi}$

≈ 0.682

$= 68.2\%$

15. (a) $V = \left(\dfrac{1}{{}_{47}C_5(27)}\right)(12{,}000{,}000) + \left(1 - \dfrac{1}{{}_{47}C_5(27)}\right)(-1)$

$\approx -\$0.71$

(b) $V = \dfrac{1}{36}(1) + \dfrac{1}{36}(4) + \dfrac{1}{36}(9) + \dfrac{1}{36}(16) + \dfrac{1}{36}(25) + \dfrac{1}{36}(36) + \dfrac{30}{36}(0) \approx 2.53$

$\dfrac{60}{2.53} \approx 24$ turns

Chapter 9 Practice Test

1. Write out the first five terms of the sequence $a_n = \dfrac{2n}{(n+2)!}$.

2. Write an expression for the nth term of the sequence $\frac{4}{3}, \frac{5}{9}, \frac{6}{27}, \frac{7}{81}, \frac{8}{243}, \ldots$

3. Find the sum $\sum_{i=1}^{6}(2i - 1)$.

4. Write out the first five terms of the arithmetic sequence where $a_1 = 23$ and $d = -2$.

5. Find a_n for the arithmetic sequence with $a_1 = 12$, $d = 3$, and $n = 50$.

6. Find the sum of the first 200 positive integers.

7. Write out the first five terms of the geometric sequence with $a_1 = 7$ and $r = 2$.

8. Evaluate $\sum_{n=1}^{10} 6\left(\frac{2}{3}\right)^{n-1}$.

9. Evaluate $\sum_{n=0}^{\infty}(0.03)^n$.

10. Use mathematical induction to prove that $1 + 2 + 3 + 4 + \cdots + n = \dfrac{n(n+1)}{2}$.

11. Use mathematical induction to prove that $n! > 2^n$, $n \geq 4$.

12. Evaluate ${}_{13}C_4$.

13. Expand $(x + 3)^5$.

14. Find the term involving x^7 in $(x - 2)^{12}$.

15. Evaluate ${}_{30}P_4$.

16. How many ways can six people sit at a table with six chairs?

17. Twelve cars run in a race. How many different ways can they come in first, second, and third place? (Assume that there are no ties.)

18. Two six-sided dice are tossed. Find the probability that the total of the two dice is less than 5.

19. Two cards are selected at random form a deck of 52 playing cards without replacement. Find the probability that the first card is a King and the second card is a black ten.

20. A manufacturer has determined that for every 1000 units it produces, 3 will be faulty. What is the probability that an order of 50 units will have one or more faulty units?